NEW SYLLABUS according to National Education Policy

SEMESTER - I

Bengaluru University and Bengaluru City University

Syllabus

CA-C1T: DISCRETE STRUCTURES

Total Teaching Hours: 48 **No. of Hours / Week: 03**

UNIT – I **[12 Hours]**

Set Theory and Logic: Fundamentals of Set theory, Set Operations and the Laws of Set Theory, Counting and Venn Diagrams, Cartesian Products and Relations, Functions–One-to-One, Onto Functions, Function Composition and Inverse Functions. Mathematical Induction, The well ordering principle, Recursive Definitions, Structural Induction, Recursive algorithms. Fundamentals of Logic, Propositional Logic, Logical Connectives and Truth Tables, Logic Equivalence, Predicates and Quantifiers.

UNIT - II **[12 Hours]**

Counting and Relations: Basics of counting, Pigeonhole Principle, Permutation and Combinations, Binomial coefficients. Recurrence relations, Modeling with recurrence relations with examples of Fibonacci numbers and the tower of Hanoi problem. Divide and Conquer relations with examples (no theorems). Definition and types of relations, Representing relations using matrices and digraphs

UNIT - III **[12 Hours]**

Matrices: Definition, order of a matrix, types of matrices, operations on matrices, determinant of a matrix, inverse of a matrix, rank of a matrix, linear transformations, applications of matrices to solve system of linear equations.

UNIT – IV **[12 Hours]**

Graph Theory: Graphs: Introduction, Representing Graphs, Graph Isomorphism, Operations on graphs. Trees: Introduction, Applications of Trees, Tree Traversal, Spanning Trees, Minimum Spanning Trees, Prim's and Kruskul's Algorithms. Connectivity, Euler and Hamilton Paths, Planar Graphs. Directed graphs: Fundamentals of Digraphs, Computer Recognition-Zero-One Matrices and Directed Graphs, Out-degree, in-degree, connectivity, orientation, Eulerian and Hamilton directed graphs, tournaments.

Unit II : BASICS OF COUNTING — 2

2.0 Basics of counting:

In many situations of computational work, the techniques of counting is essential. The two basic rules of counting are **'sum rule'** and the **'product rule'**.

2.1

1. **The sum rule: (Fundamental principle of addition)** Suppose E_1 and E_2 be two events, to be performed. If the event E_1 can be performed in 'm' different ways and the event E_2 can be performed in 'n' different ways and if these two events cannot be performed simultaneously, then one of the two events (E_1 or E_2) can be performed in (m + n) ways.

   In general if the events $E_1, E_2, E_3, \ldots, E_n$ are 'n' events can happen respectively in $m_1, m_2, m_3, \ldots m_n$ ways such that no two of these events can be performed at the same time, then the number of ways of the happening of one of the events (E_1 or E_2 or E_3 ... or E_n) can be performed in $(m_1 + m_2 + m_3 + \ldots + m_n)$ different ways.

Illustrative Examples:

Example 1: In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make the selection.

Solution:

The teacher is to perform either of the following two operations

(i) Selecting a boy among 10 boys and

(ii) Selecting a girl among 8 girls

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by sum rule either of the two operations can be performed in (10 + 8) = 18 ways.

Example 2: A small library has 15 books on Mathematics, 12 books on Physics 10 books on Electronics and 14 books on Computer Science. A student wishes to choose one of the books for study. The number of ways the student can choose a book from the library is *(15 + 12 + 10 + 4) = 41 ways.*

2.2

2. **The product rule:** Let E_1 and E_2 be two events which are to be performed one after the other. If E_1 can be performed in 'm' different ways and for each of these ways E_2 can be performed in 'n' different ways, then both of the events can be performed in mn different ways.

   In general if the events $E_1, E_2, E_3, \ldots, E_n$ are such that E_1 happens in m ways, E_2 happens in m_2 ways, ..., E_n happens in m_n ways. Then the event E_1 followed by E_2 followed by E_3... followed by E_n in a sequential order happens in $m_1, m_2, m_3 \ldots m_n$ ways.

Illustrative Examples:

Example 1: Suppose a cricket ground has 3 entrance gates and 4 exit gates. A person can enter and leave the ground in $3 \times 4 = 12$ different ways.

Example 2: There are 6 shirts and 5 pants with a student. In how many ways can he go to the college with different combination.

Solution:

With each shirt the student can wear any one of the 5 pants resulting in a different combination. Since there are 6 shirts in all, the number of different combinations $= 6 \times 5 = 30$.

Example 3: Suppose a boarding hotel sells 5 south Indian dishes, 3 North Indian dishes, 3 hot beverages and 2 cold beverages. For breakfast, a person wishes to buy 1 south Indian dish and 1 hot beverage or 1 North Indian dish and 1 cold beverage.

Solution:

The person can have the first choice in $5 \times 3 = 15$ ways and the he can have the second choice on $3 \times 2 = 6$ ways.

The total number of ways the person can buy his breakfast in $15 + 6 = 21$ ways.

Example 4: A bit is either 0 or 1. A byte is a sequence of 8 bits. Find (i) the number of bytes (ii) the number of bytes that begin with 1 1 and end with 1 1.

Solution:

(i) Given a byte contains 8 bits and each bit is 0 or 1 (two choices), the number of bytes is $2^8 = 256$

(ii) In a byte beginning and ending with 11, there occur 4 open positions. These can be filled in $2^4 = 16$ ways. Therefore there are 16 bytes which begin and end with 1 1.

Example 5: How many 3 digit numbers can be formed from the digits 1, 2, 3, 4, and 5 assuming that (i) repetition of the digits is allowed.

(ii) repetition of the digits is not allowed.

Solution:

(i) Repetition of the digits is allowed.

There are 5 digits namely, 1, 2, 3, 4 and 5.

5	5	5

Every digit can be selected any number of times. Hence we can select first digit in 5 ways, the second digit in 5 ways and the third digit in 5 way. By product rule total number of possibilities of forming a 3 digit number is $5 \times 5 \times 5 = 125$ ways.

(ii) Repetetion of the digits is not allowed:

Under the restriction, the first digit can be selected in 5 ways after the selection of the first digit second digit can be selected in 4 ways and the third digit in 3 ways. By product rule, total number of three digit numbers = $5 \times 4 \times 3 = 60$ ways.

EXERCISE:

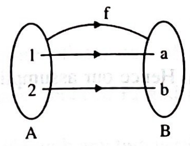
1. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there.
2. How many four digit numbers can be formed using the digits 1, 3, 5, 8 and 9, the repetition of the digits are not allowed.
3. Find the number of (i) 2 digits even numbers (ii) 2 digit odd numbers.
4. How many 3-digits numbers can be formed by using the digits 2, 3, 4, 5, 6, 8 if
   (i) repetetions of the digits are allowed.
   (ii) repetetions of the digits are not allowed.

ANSWERS:

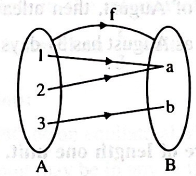
(1) 8 *(2) 120* (3) (i) 45 (ii) 45 (4) (i) 216 (ii) 120

2.3 The Pigeonhole principle:

Recall: Let A and B, be two non empty sets, with $|A| = m$, $|B| = n$. If $|A| = |B|$ then $f: A \to B$ has a one-one correspondence. (one-one correspondence means the function is both one-one and onto)



If $|A| > |B|$, then $f: A \to B$ is such that at least two different elements of A have the same image in B.



i.e.,

This concept which is already discussed leads to the "Pigeonhole principle."

The Pigeonhole principle:

If 'm' Pigeons occupay 'n' Pigeonholes and if m > n then atleast one Pigeonhole contains two or more Pigeons in it.

Generalization:

The following is an extension/generalization of the Pigeonhole principle.

If 'm' Pigeons occupy n Pigeonholes m > n then atleast one Pigeonhole contains $\left(\frac{m-1}{n}\right)+1$ Pigeons.

Proof: We prove this principles by the method of contradiction.

Assume that no Pigeonhole contains $\left(\frac{m-1}{n}\right)+1$ or more Pigeons. This means that every Pigeonhole contains $\left(\frac{m-1}{n}\right)$ or less number of Pigeons.

Then, total number of Pigeons $\leq n\left(\frac{m-1}{n}\right)=(m-1)$

Which is a contradiction, because the total number of Pigeons is m. Hence our assumption is wrong and the principle is true.

As a simple application of the above principle, note that

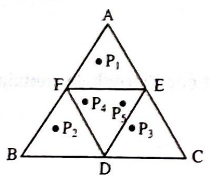
(i) if 8 children are born in the same week, then two or more children are born on the same day of the week.

(ii) if 32 children in a school have their birthdays in the month of August, then atleast two children will have their birthday on the same date of the month as August has 31 days.

WORKED EXAMPLES:

Example 1: Let ABC be an equilateral triangle whose sides are of length one unit. Show that if any five points are chosen lying on or inside the triangle then two of them must be not more than $\frac{1}{2}$ unit apart.

Solution:



Let ABC be an equilateral triangle. D, E, F are mid points of BC, AC and AB respectively. The lines DE, EF and FD divides the triangles ABC into four equilateral triangles with each side $\frac{1}{2}$ unit since AB = BC = CA = 1 unit.

Now applying Pigeonhole principle

Number of points (m) = 5 (number of Pigeons)

Number of equilateral triangles (n) = 4 (number of Pigeonholes)

According to the generalized Pigeonhole principle one of the Pigeonholes must contain atleast $\left(\frac{m-1}{n}\right)+1$ Pigeons. i.e., $\left(\frac{5-1}{4}\right)+1=2$ Pigeons

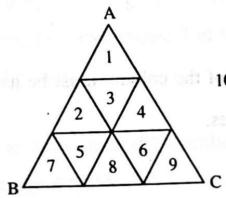
Therefore atleast one of the four small triangles contains atleast two points. (P_4 and P_5) Distance between P_4 and P_5 is less than $\frac{1}{2}$ unit and it will be equal to $\frac{1}{2}$ unit if P_4 and P_5 coincide with any two vertices of the triangles DEF. Hence the result.

Example 2: Let ABC be an equilateral triangle of side 1 unit. Show that if we select 10 points in the interior of the triangle, there must be atleast two points whose distance apart is less than $\frac{1}{3}$ unit.

Solution:

Let ABC be an equilateral triangle each side is equal t 1 unit. i.e., AB = BC = CA = 1 unit.

10^{th} point may be in any of the 9 triangles



10^{th} point may be in any of the 9 triangles

AB, BC, CA are trisected to form 9 equilateral triangles, with each side is equal to $\frac{1}{3}$ unit.

Number of points = m = 10 = Number of Pigeons

Number of small equilateral triangles = n = 9 = Number of Pigeonholes.

Applying the generalized Pigeonhole principle one of the Pigeonhole must contain $\left(\frac{m-1}{n}\right)+1$ Pigeons.

i.e., $\left(\frac{10-1}{9}\right)+1=2$ Pigeons

Therefore atleast one of the 9 small triangles contain atleast two points. The distance between two points is less than $\frac{1}{3}$ unit, since the length of any side of the small triangle is $\frac{1}{3}$ unit.

Example 3: If 9 colours are used to paint 100 doors, show that atleast 12 doors will have the same colour.

Solution:

By data, number of doors = 100 = m = number of Pigeons.

Number of colours used = 9 = n = Number of Pigeonholes.

Using the generalized Pigonhole principle that atleast one of the colours must be assigned to

$\left(\frac{m-1}{n}\right)+1=\left(\frac{100-1}{9}\right)+1=\left(\frac{99}{9}+1\right)=12$ or more doors.

Example 4: If 5 colours are used to paint 26 houses prove that atleast 6 houses will have the same colour.

Solution:

Number of houses = m = 26 = Number of Pigeons

Number of colours used = n = 5 = Number of Pigeon holes.

Using the generalized Pigeonhole principle that atleast one of the colours must be assigned to

$\left(\frac{m-1}{n}\right)+1=\left(\frac{26-1}{5}\right)+1=\left(\frac{25}{5}+1\right)=5+1=6$ or more houses.

Example 5: Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any 3 of these students are chosen to be a debating team from the class, the sum of their shirt numbers is used as the code number of the team. Show that if any 8 of the 20 are selected, then from these 8 we may form atleast two different teams having the same code number.

Solution:

The number of teams comprising 3 students out of 8 selected is equal to

$$8C_3=\frac{8!}{(8-3)!\,3!}=\frac{8!}{5!\,3!}\text{ i.e., }\frac{8\times7\times6\times5!}{5!(3\times2\times1)}=56$$

According to the way in which the code number of a team is determined, note that the smallest possible code number is 1 + 2 + 3 = 6 and the largest possible code number is 18 + 19 + 20 = 57. Thus the code numbers vary from 6 to 57. The set comprising code numbers = {6, 7, 8, ...57} = 52. Possible number of teams = m = 56 = Number of Pigeons. Possible number of code numbers = n = 52 = Number of Pigeonholes.

Using the generalized Pigonhole principle one of the Pigeonholes must contain atlest $\left(\frac{m-1}{n}\right)+1$ Pigeons i.e., $\left(\frac{56-1}{52}\right)+1=(1.06)+1=1+1=2$

Thus we observe that atleast 2 different teams will have the same code.

Example 6: An office employs 13 clerks. Show that atleast two of them will have birthdays during the same month of year

Solution:

Clerks in the office = m = 13 number of pigeons. Months in the year = n = 12 = number of Pigeonholes. Using the generalized Pigeonhole principle one of the pigeonholes must contain atleast $\left(\frac{m-1}{n}\right)+1=\left(\frac{13-1}{12}\right)+1=\frac{12}{12}+1=1+1=2$

Thus we conclude that atleast 2 of the Clerks in the office will have birthdays during the same month of the year.

Example 8: Show that if 7 numbers are selected from 1 to 12, then two of them will have their sum equal to 13.

Solution:

Let A = {1, 2, 3, ..., 12}. Two numbers from A adding to 13 are the sets

$A_1=\{1,12\}$, $A_2=\{2,11\}$, $A_3=\{3,10\}$, $A_4=\{4,9\}$, $A_5=\{5,8\}$, $A_6=\{6,7\}$

These are the only sets containing two numbers from 1 to 12 whose sum is 13.

Since every number from 1 to 12 belongs to one of the above sets, each of the seven numbers (6 + 1) chosen must belong to one of the six sets. Evidently two of the seven chosen numbers should belong to the same set having their sum equal to 13 (according to the Pigeonhole principle).

Example 9: Show that if any (n + 1) numbers from 1 to 2n are chosen, then two of them will have their sum equal to (2n + 1)

Solution:

Let A = {1, 2, 3, …, (n − 1), n, (n + 1), …, 2n}. Two numbers from A adding to (2n + 1) are the sets,

$A_1 = \{1, 2n\}, A_2 = \{2, 2n-1\}, A_3 = \{3, 2n-2\}\ldots$

$A_{n-1} = \{n-1, n+2\}, A_n = \{n, n+1\}$

These are the only sets containing two numbers from 1 to 2n whose sum is (2n + 1).

Since every number from 1 to 2n belongs to one of the above sets, each of the (n + 1) numbers chosen must belong to one of the sets. Since there are only 'n' sets, two of the (n + 1) chosen numbers have to belong to the same set according to the Pigeonhole principle. These two numbers have their sum equal to (2n + 1).

EXERCISE:

1. Show that among one lakh people atleast two people are born at the same hour, minute and second.
2. Six books each of Physics, Computer science, Mathematics and four books of Economics totally contains 12225 pages. Find the least number of pages contained in a book.

   Ans: 556
3. Show that a research paper in Mathematics subject having 48 pages comprising 5722 words, one of the page should have atleast 120 words.
4. Prove that in a set of 13 children atleast two have birthdays during the same month.
5. If seven cars carry 26 passengers, prove that atleast one car must have 4 or more passengers.
6. Prove that if 30 dictonaries in a library contain a total of 61,327 pages, then atleast one of the dictionaries must have atleast 2045 pages.
7. Prove that in any set of 29 persons atleast five persons must have been born on the same day of the week.
8. *Show that if any 6 numbers from 1 to 10 are chosen, then two of them have their sum equal to 11.*

9. A bag contains many red marbles, many white marbles and many black marbles. What is the least number of marbles one should take out to be sure of getting atleast six marbles of the same colour.

   Ans: 16
10. Show that if 50 books in a library contain a total of 37551 pages, one of the books must have atleast 752 pages.

2.4 Permutations and Combinations:

Introduction: The concept of **'Permutations and Combination'** can be traced back to Vedic period. The credit however goes to the Jains who treated its subject matter as a self contained topic in Mathematics under the name **'Vikalpa'**. Among the Jains **'Mahavira'** (850 A.D) is perhaps the worlds first mathematician credited with providing a general formula for Permutations and Combinations.

In the 6th century **'B.C. Sushruta'** in his medicinal work **'Sushruta Samhita'** asserts that 63 combinations can be made out of 6 different tastes taken one at a time, two at a time etc. **'Pingala'**, a Sanskrit scholar around 3rd century B.C. gives the method of determining the number of combinations of a given number of letters taken one at a time, two at a time etc in his work **'Chhanda Sutra'. 'Bhaskara charya' (1114 A.D)** treated the subject matter of **'Permutations and Combinations'** under the name **'Ankapash'** in his famous work **'Lilavathi'**. He has given several important theorems and results in addition to the general formula for ^nP_r and ^nC_r, already provided by **'Mahavira'**. Swiss mathematician **'Jacob Bernouli' (1654 – 1705 A.D)** has given a complete treatment of **'Permutations and Combinations'** in his book **'Ars conject andhi'** (Posthumously published in 1713 A.D)

Factorial Notation:

The continued product of first 'n' natural numbers is called **'factorial n'** and is denoted by the symbol n! or $\underline{|n}$.

Thus $n! = 1\cdot 2\cdot 3\cdot\ldots(n-1)n$ or n! = n(n − 1) (n −2) …3.2.1.

Deductions: 1 = 1! $1\times 2 = 2!$ or $1\cdot 2 = 2!$ $1\cdot 2\cdot 3\cdot 4 = 4!$

Note: By definition 0! = 1

In particular $5! = 5\cdot 4! = 5\cdot 4\cdot 3! = 5\cdot 4\cdot 3\cdot 2! = 5\cdot 4\cdot 3\cdot 2\cdot 1$

For a natural n,

$$n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$$

Factorial is defined only for whole numbers.

WORKED EXAMPLES:

Example 1:

Evaluate (i) 4! **(ii) 6!** **(iii) 6! – 5!** **(iv) $\frac{8!}{6!\ 2!}$**

Solution:

(i) $4! = 4\cdot 3\cdot 2\cdot 1 = 24$

(ii) $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

(iii) $6! - 5! = 720 - 120 = 600$

(iv) $\frac{8!}{6!2!} = \frac{8\times 7\times 6!}{6!2\cdot 1} = \frac{56}{2} = 28$

Example 2:

(i) If $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$ find x

(ii) Is $3! + 4! = 7!$

Solution:

(i) $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$

$\Rightarrow \frac{8\cdot 7}{8\cdot 7\cdot 6!} + \frac{8}{8\cdot 7!} = \frac{x}{8!}$

$\Rightarrow \frac{56}{8!} + \frac{8}{8!} = \frac{x}{8!}$

$\Rightarrow \frac{56+8}{8!} = \frac{x}{8!} \Rightarrow x = 64$

(ii) $3! + 4! \neq 7!$

Because LHS $3\cdot 2\cdot 1 + 4\cdot 3\cdot 2\cdot 1 = 6 + 24 = 30$

RHS $7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$ LHS $\neq$ RHS

2.5 Permutations:

Suppose we have three objects A, B and C, we are asked to select two objects at a time. The selections are AB, BC and AC. Thus the number of ways of selecting two objects out of the three given objects is 3. Now each of these selection can be arranged in two different ways and we get the following arrangement, AB, BA; BC, CB; AC, CA.

Thus we get six arrangements.

So the number of arrangements of three different objects taken two at a time is 6. From the above discussion, we came to know that we have two types of classification of objects or things namely

(i) Selection and (ii) arrangements

Selection of objects is called combinations and the **arrangements** of objects is called permutations.

First we shall study about **'permutations'** then go to **'combinations'.**

Definition: A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

Note: We use the symbol ^nP_r or P(n, r) to denote the number of permutations of 'n' different things taken 'r' at a time. $(n \in N, 0 < r \leq n)$

Formula for ^nP_r:

Permutations when all the objects are distinct:

Find the number of permutations of 'n' different objects taken 'r' at a time. $(0 < r \leq n)$ (and the objects do not repeat)

We have 'n' distinct objects and wish to arrange 'r' of these objects in a line. Since there are 'n' ways of choosing the first object and after this is done, we are left with (n – 1) objects. Now the second object can be chosen in (n –1) ways, third in (n – 2) ways and so on and finally (n – (r –1)) ways of choosing the r^{th} object. Now by product rule of counting (studied in the preceding section) that the number of different arrangements or permutations is n (n – 1) (n – 2)... (n – (r – 1)). This number is denoted by ^nP_r or P(n, r).

This expression can be modified as follows:

$$^nP_r = n(n-1)(n-2)\ldots(n-(r-1))$$

$$= \frac{n(n-1)(n-2)\ldots(n-(r-1))(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1} \quad \therefore {}^{n}P_r = \frac{n!}{(n-r)!} = P(n,r)$$

In particular, if $r = n$ then ${}^{n}P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! (\because 0! = 1)$

Thus the number of different arrangements of 'n' distinct objects taken all a time is n!

WORKED EXAMPLES:

Example 1: Evaluate (i) ${}^{20}P_4$ (ii) P(16, 5) (iii) P(5, 5) (iv) ${}^{8}P_5$

Solution:

(i) $${}^{20}P_4 = \frac{20!}{(20-4)!} = \frac{20!}{16!} = \frac{20\times19\times18\times17\times16!}{16!} = 20\times19\times18\times17$$

$= 116280.$

(ii) $$P(16,5) = \frac{16!}{(16-5)!} = \frac{16!}{11!} = \frac{16\times16\times14\times13\times12\times11!}{11!} = 16\times15\times14\times13\times12 = 524160$$

(iii) $$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$$

(iv) $${}^{8}P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3!}{3!} = 6720$$

Example 2: Find r, if ${}^{5}P_r = 2\times {}^{6}P_{r-1}$

Solution:

We have ${}^{5}P_r = 2\cdot {}^{6}P_{r-1}$

$$\Rightarrow \frac{5!}{(5-r)!} = 2\cdot\frac{6!}{(6-(r-1))!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = 2\cdot\frac{6!}{(6-r+1)!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = 2\cdot\frac{6\times5!}{(7-r)!}$$

$$\Rightarrow \frac{1}{(5-r)!} = \frac{12}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow (7-r)(6-r) = 12$$

$$\Rightarrow r^2 - 13r + 30 = 0 \Rightarrow (r-10)(r-3) = 0 \Rightarrow r = 10, 3\ r \neq 10, \therefore r = 3$$

Example 3: If ${}^{n-1}P_3 : {}^{n}P_4 = 1:9$ find n

Solution:

Given $\frac{{}^{n-1}P_3}{{}^{n}P_4} = \frac{1}{9}$

$$\Rightarrow \frac{\frac{(n-1)!}{(n-4)!}}{\frac{n!}{(n-4)!}} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n!} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n(n-1)!} = \frac{1}{9} \Rightarrow \frac{1}{n} = \frac{1}{9} \Rightarrow n = 9$$

Example 4: Establish the following identities.

(i) ${}^{n}P_r = n\times {}^{n-1}P_{r-1}$ **(ii) ${}^{n}P_r = {}^{n-1}P_r + r\times {}^{n-1}P_{r-1}$**

(iii) ${}^{(n+1)}P_r = (n+1)\times {}^{n}P_{r-1}$ **(iv) ${}^{n}P_r = (n-r+1)\times {}^{n}P_{r-1}$**

Solution:

(i) ${}^{n}P_r = n\cdot {}^{n-1}P_{r-1}$

LHS ${}^{n}P_r$

$$\text{RHS } n\cdot {}^{n-1}P_{r-1} = n\cdot\frac{(n-1)!}{(n-1-(r-1))!}$$

$$= \frac{n(n-1)!}{(n-1-r+1)!} = \frac{n!}{(n-r)!} = {}^{n}P_r \therefore \text{LHS} = \text{RHS}.$$

(ii) ${}^{n}P_r = {}^{n-1}P_r + r\cdot {}^{n-1}P_{r-1}$

LHS $= {}^{n}P_r$

$$\text{RHS} = \frac{(n-1)!}{(n-1-r)!} + r\cdot\frac{(n-1)!}{(n-1-r+1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\cdot\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\cdot\frac{(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[1 + \frac{r}{n-r}\right] = \frac{(n-1)!}{(n-r-1)!}\left[\frac{n-r+r}{n-r}\right]$$

$$= \frac{n(n-1)!}{(n-r)(n-r-1)!} = \frac{n!}{(n-r)!} = {}^{n}P_r = \text{LHS}$$

(iii) ${}^{(n+1)}P_r = (n+1)\cdot {}^nP_{r-1}$

$$LHS\ {}^{(n+1)}P_r = \frac{(n+1)!}{(n+1-r)!} = \frac{(n+1)!}{(n-r+1)!}$$

$$= (n+1)\frac{n!}{(n-r+1)!} = (n+1)\ {}^nP_{r-1} = RHS$$

(iv) ${}^nP_r = (n-r+1)\cdot {}^nP_{r-1}$

$$RHS = (n-r+1)\frac{n!}{(n-r+1)!} = \frac{(n-r+1)n!}{(n-r+1)(n-r)!} = \frac{n!}{(n-r)!} = {}^nP_r = LHS$$

Example 5: How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated.

Solution:

Required number of 3-digit numbers is equal to ${}^9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = \frac{9\cdot 8\cdot 7\cdot 6!}{6!} = 9\cdot 8\cdot 7 = 504$

Example 6: How many 4-digit numbers are there with no digit repeated, using the digits 0, 1, 2, ...9.

Solution:

Number of arrangements of 10 digits taken 4 at a time is ${}^{10}P_4$. This includes arrangements where the digit 0 is at the 1000th place, but such numbers are 3-digit numbers, which are to be excluded from the total numbers. ∴ Required 4-digit numbers

$$= {}^{10}P_4 - {}^9P_3 = \frac{10!}{6!} - \frac{9!}{6!} = \frac{10! - 9!}{6!} = \frac{10\times 9! - 9!}{6!} = \frac{9!(10-1)}{6!} = \frac{9\times 9!}{6!}$$

$$= \frac{9\times 9\times 8\times 7\times 6}{6!} = 9\times 9\times 8\times 7 = 4536$$

Example 7: Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these are even.

Solution:

We have to form 4-digit numbers using the given 5 digits. This can be done in ${}^5P_4$ ways

i.e., $\frac{5!}{(5-4)!} = 5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

5	4	3	2

Even 4 digit numbers:

A number is even if the digit in the unit place is 2, 4

Numbers ending with 2 is $4\cdot 3\cdot 2 = 24$

4	3	2	
			2

Numbers ending with 4 is $4\cdot 3\cdot 2 = 24$

4	3	2	
			4

Therefore total number of 4 digit even numbers are 24 + 24 = 48

Example 8: A family of 4 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated (i) All the sisters sit together (ii) No two sisters sit together.

Solution:

(i) All the sisters sit together

Consider 3 sisters as one unit, remaining 4 brothers, in total 1 + 4 = 5 different units taken 5 at a time in a row. This can be arranged in ${}^5P_5 = 5!$ ways. Among themselves the 3 sisters can be arranged in ${}^3P_3$ ways = 3! ways.

∴ The required number of ways = 5! × 3! = 120 × 6 = 720 ways

(ii) No two sisters sit together

Since there is no condition on 4 brothers they can be arranged in ${}^4P_4 = 4!$ Ways

✓ B_1 ✓ B_2 ✓ B_3 ✓ B_4 ✓

Now there are 5 places vacant in which the three sisters can be seated so that no two sisters are together. This can be done in ${}^5P_3$ ways.

Hence the total number of arrangements

$$= 4!\times {}^5P_3 = 4!\times\frac{5!}{2!} = \frac{24\times 120}{2} = 24\times 60 = 1440 \text{ ways}$$

Example 9: In how many ways 4 mathematics books, 5 Physics books, 4 Chemistry books and 3 Biology books can be arranged in a shelf so that all books of the same subject are together.

Solution:

Considering books of a particular subject as one unit, there are 4 units and they can be arranged in ${}^4P_4 = 4! = 24$ ways. Now among themselves Mathematics books can be arranged in 4! ways, Physics books can be arranged in 5! ways, Chemistry books in 4! ways and Biology books in 3! ways.

Hence the total number of arrangements

$= 4! \times 4! \times 5! \times 4! \times 3!$ ways

$= 24 \times 24 \times 120 \times 24 \times 6 = 9953280$ ways

EXERCISE:

1. How many words with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated if
   (i) 4 letters are used at time
   (ii) All letters are used at a time
   (iii) All letters are used but first letter is a Vowel.
2. Find r, if $5 \cdot {}^4P_r = 6 \cdot {}^5P_{r-1}$
3. Find n, if ${}^{(2n+1)}P_{n-1} : {}^{(2n-1)}P_n = 3:5$
4. Find n, if ${}^nP_3 = 13 \cdot {}^{(n+1)}P_3$
5. How many numbers greater than 56000 can be formed by using the digits 4, 5, 6, 7, 8 no digit being repeated in any number.
6. Find the number of ways in which 5 boys and 5 girls be seated in a row so that
   (i) No two girls may sit together
   (ii) All the girls sit together and all the boys sit together.
7. In how many ways can the letters of the word PENCIL be arranged so that
   (i) N is always next to E
   (ii) N and E are always together.

8. The letters of the word RANDOM are written in all possible orders and these words are written out as in dictionary. Find the rank of the word RANDOM.
9. How many words with or without meaning of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE.

ANSWERS:

1. (i) 360 (ii) 720 (iii) 240
2. $r = 3 \ (r \neq 8)$
3. $n = 4 \left(n \neq -\frac{1}{3} \right)$
4. 15
5. 90
6. $5! \times 6!$
7. (i) 120 (ii) 240
8. 614
9. 288

2.6 Permutations when all the objects are not distinct objects:

The number of permutations of 'n' objects taken all at a time when 'n_1', of them are alike of one type, 'n_2' of them are alike of second type … 'n_k' of them are alike of k^{th} type where $n_1 + n_2 + n_3 + \ldots + n_k = n$ then the number of permutations of 'n' things by taking all the things at a time is given by $\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_k!}$

Proof: Let 'x' be the number of such permutations. Each of these x permutations contains 'n_1' like objects of first type. Replace 'n_1' like objects by 'n_1' distinct objects. These 'n_1' distinct objects can be arranged among themselves in $n_1!$ Ways. Thus from each of the 'x' permutations we get $n_1!$ new permutations. When this is done in all the x permutations we get $xn_1!$ new permutations. Each of these $xn_1!$ permutations contain only 'n_2' like objects and the rest distinct.

Replace these 'n_2' like objects by 'n_2' distinct objects. These 'n_2' distinct objects can be arranged among themselves in $n_2!$ ways. Therefore from each of the $xn_1!$ Permutations, we get $n_2!$ new permutations. Therefore from all the $xn_1!$ Permutations, we get $xn_1!\,n_2!$ permutations.

continuing this finally we have $xn_1!\,n_2!\,n_3!\ldots n_k!$ permutations. Each of these $xn_1!\,n_2!\,n_3!\ldots n_k!$ permutations have all the n objects distinct. But the number of permutations of 'n' distinct objects taken all at a time is n! $\therefore xn_1!n_2!n_3!\ldots n_k! = n!$ or $x = \frac{n!}{n_1!\,n_2!\,n_3!\ldots n_k!}$.

WORKED EXAMPLES:

Example 1: In how many ways can the letters of the following word be arranged

(i) ALLAHABAD (ii) ASSAM

Solution:

(i) In the word ALLAHABAD, we have A–4, L–2, the remaining letters are all distinct. Total number of letters n = 9. $\therefore$ Required number of permutations $= \frac{9!}{4!\,2!}$

(ii) In the word ASSAM, we have A–2, S–2 the remaining letter is distinct total number of letters n = 5. $\therefore$ Required number of permutations $= \frac{5!}{2!\,2!} = 30$

Example 2: Find the number of permutations of the letters of the word. ENGINEERING. In how many of these (i) all the three E's together (ii) how many of them begin with N.

Solution:

The given word has 11 letters of which E-3, N-3, I-2, G-2, R-1.

Number of permutations of the letter of the word ENGINEERING, without any condition is $\frac{11!}{3!\,3!\,2!\,2!}$

(i) All the three E's together:

Treat the 3 E's as one letter, i.e., E – 1, N – 3, I – 2, G – 2, R – 1

$\therefore$ Required number of permutations $= \frac{9!}{3!\,2!\,2!} = 15120$

(ii) Arrangements begin with N:

For arrangements beginning with N, Fix one N in the first place, the remaining letters are E–3, N–2, I–2, G–2, R–1. $\therefore$ The required number of permutations $= \frac{10!}{3!\,2!\,2!\,2!} = 75600$

Example 3: In how many different ways can the letters of the word MISSISSIPPI be arranged. How many of these (i) has all the I's together (ii) words start with M and end with I.

Solution:

In the given word there are 11 letters, out of which M-1, I-4, S-4, P-2.

Total number of arrangements without any restriction is $\frac{11!}{4!\,4!\,2!} = 34650$

(i) All the I's together:

Treat all the 4 I's as one, the remaining M-1, S-4, P-2. In total 1 + 7 = 8

$\therefore$ Required number of permutations $= \frac{8!}{4!\,2!} = 840$

(ii) Words start with M and end with I.

Fix M in the first place and one I in the end, the remaining letters are I-3, S-4, P-2

Hence required number of permutations $= \frac{9!}{3!\,4!\,2!} = 7560$

Example 4: How many positive integers 'n' can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want 'n' to exceed 5,000,000.

Solution:

The number of digits are 7 in number. i.e., 'n' is of the form $n = d_1 d_2 d_3 d_4 d_5 d_6 d_7$. If 'n' has to exceed 5,000,000 which also have 7 digits, then it is necessary that d_1 is 5 or 6 or 7.

Case (i): Suppose $d_1 = 5$, the rest of the digits are 3 – 1, 4 – 2, 5 – 1, 6 – 1, 7 – 1

$\therefore$ The number of permutations $= \frac{6!}{2!} = 360$

Case (ii): Suppose $d_1 = 6$, the rest of the digits are 3 – 1, 4 – 2, 5 – 2, 7 – 1

$\therefore$ The number of permutations $= \frac{6!}{2!\,2!} = 180$

Case (iii): Suppose $d_1 = 7$, the rest of the digits are 3 – 1, 4 – 2, 5 – 2, 6 – 1.

$\therefore$ The number of permutations $= \frac{6!}{2!\,2!} = 180$

By using sum rule, the required number of positive integers that exceeds 5,000,000 is 360 + 180 + 180 = 720

EXERCISE:

1. In how many ways can the letters of the word 'PERMUTATIONS' be arranged if the
   (i) words start with P and end with S
   (ii) Vowels are all together.
2. Find the number of permutations of the letter of the word 'MASSASAUGA'. In how many of these all the four A's together? How many of them begin with S.
3. How many positive integers 'n' can are form using the digits 3, 4, 4, 5, 5, 6, 7 if we want 'n' to exceed 5,000,000.
4. How many numbers greater than 100000 can be formed using the digits 1, 2, 2, 2, 4, 4, 0.
5. In how many ways can the letters of the word 'ASSASSINATION' be arranged so that all the S's are together
6. Find the number of permutations of the letters of the words
   (i) MATHEMATICS (ii) DISCRETE
   (iii) STRUCTURES (iv) CONSTANTIONOPLE

ANSWERS:

1. $\frac{12!}{2!}, \frac{10!}{2!}$, 2419200
2. 25,200,840,7560
3. 720
4. 360
5. 151200
6. (i) 4989600 (ii) 20160
   (iii) 226800 (iv) $\frac{14!}{24}$

2.7 COMBINATIONS

In the previous section, we studied different permutations of the objects, we found that the order of occurrence of the objects was important. We now consider selection of the required number of objects out of the total number of objects without regard to order within the selection.

For example: Suppose out of 3 letters A, B and C we have to select 2.

Then, the following are the possible different choices (i) AB (ii) BC (iii) CA

In the above selection AB and BA, BC and CB, CA and AC are same. Hence out of 3 things, 2 things can be selected in 3 ways. A combination is a selection of some or all of a number of different objects. In a combination the order of appearance of the objects is immaterial.

The difference between a permutation and combination of the objects is that 'order does matter in a permutation, while it does not matter in the case of a combination.

In the above example, we have seen that there are 3 different combinations of A, B, C taken 2 at a time. If we denote the number of combinations of 3 object taken 2 at a time by ${}^3C_2$ then we have ${}^3C_2 = 3$.

We also know the number of different permutations of 3 objects taken 2 at a time is

$${}^3P_2 = \frac{3!}{(3-2)!} = 3\times2\times1 = 6$$

Note that each combination gives rise to 2 permutations, for instance combination like AB gives rise to two permutations AB and BA. Thus, ${}^3C_2 2! = {}^3P_2$

i.e., $${}^3C_2 = \frac{{}^3P_2}{2!}$$

To find the value of ${}^nC_r$

The number of combinations of n distinct objects taken r at a time is denoted by ${}^nC_r$

Expression for ${}^nC_r$:

${}^nC_r$ is the number of ways of selecting r objects out of n. Each combination has r objects, which can be arranged among themselves in r! ways. Thus one combination gives rise to r! permutations. ${}^nC_r$ combinations give rise to ${}^nC_r$. $\underline{|r}$ number of permutations.

But the number of permutations of n object taken 'r' at a time is ${}^nP_r$

$$\therefore \quad {}^nC_r \underline{|r} = {}^nP_r$$

$${}^nC_r = \frac{{}^nP_r}{\underline{|r}}$$

Using the result ${}^nP_r = n(n-1)(n-2)...(n-r+1)$, we get

$${}^nC_r = \frac{n(n-1)(n-2)...(n-r+1)}{\underline{|r}} \quad \text{Also } {}^nP_r = \frac{\underline{|n}}{\underline{|n-r}} \quad \therefore {}^nC_r = \frac{\underline{|n}}{\underline{|r}\cdot\underline{|n-r}}$$

Note:

(1) The number of combination of n objects taken all at a time is 1. This can easily be verified.

We have $^nC_r = \frac{n!}{r!(n-r)!}$ put r = n

$$^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$$

(2) Complimentary combination: $^nC_r = {}^nC_{n-r}$

We have $^nC_r = \frac{n!}{r!(n-r)!}$ ...(1)

Replacing r by (n − r) in (1)

We get, $^nC_r = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!(n-n+r)!}$

$$^nC_r = \frac{n!}{(n-r)!r!} \quad ...(2)$$

From (1) and (2) $^nC_r = {}^nC_{n-r}$

This result can be used to make computations of ^nC_r simple.

Note: From the above result we observe that

$^nC_x = {}^nC_y \Rightarrow x = y$ or $x = n - y$

(3) $^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$

Some important results

1. If n and r are natural numbers such that $1 \le r \le n$ then prove that $= \frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

Solution:

We know that $^nC_r = \frac{n!}{(n-r)!r!}$ ...(1)

$$^nC_{r-1} = \frac{n!}{(n-(r-1))!(r-1)!} = \frac{n!}{(n-r+1)!(r-1)!} \quad ...(2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{^nC_r}{^nC_{r-1}} = \frac{\frac{n!}{(n-r)!r!}}{\frac{n!}{(n-r+1)!(r-1)!}} = \frac{(n-r+1)!(r-1)!}{(n-r)!r!}$$

$$= \frac{(n-r+1)(n-r)!(r-1)!}{(n-r)!r(r-1)!}$$

$$= \frac{(n-r+1)}{r} \text{RHS}$$

2. Prove that $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Solution:

$$^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!(n+1)}{(r-1)!(n-r)!r(n-r+1)} = \frac{(n+1)!}{r!(n-r+1)!} = {}^{n+1}C_r = \text{RHS}$$

Note: This rule is referred as Pascal's rule. We now give an alternative proof of the above result which is based on combinatorial arguments. In fact some authors venture to call this a combinatorial proof.

We recall that the number of combinations of (n + 1) distinct objects taken r at a time is $^{n+1}C_r$. Let us concentrate on one of the given (n + 1) distinct objects and let us denote it by say S. Clearly there are two possibilities.

1) This particular object S is included in the selection

2) S is not included in the selection

When S is included in the selection, clearly all the other (r − 1) objects must be selected from the remaining [(n + 1) − 1] i.e., n objects. This can be done in $^nC_{r-1}$ ways. When S is not included in the selection all the r objects must be selected from the remaining $[(n+1)-1]$ i.e. n objects. This can be done in ^nC_r ways.

Thus the total number of ways of selection r objects out of (n +1) distinct objects is given by $^{n+1}C_r = {}^nC_{r-1}$

Note: This formula is helpful in constructing Pascal's triangle or Meru prastara.

WORKED EXAMPLES:

Example 1: Evaluate $^{19}C_{17} + {}^{19}C_{18}$

Solution:

We have $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ Put n = 19, r = 18

We get $^{19}C_{18} + {}^{19}C_{17} = {}^{20}C_{18} = {}^{20}C_{20-18}$ $(\because {}^nC_r = {}^nC_{n-r})$

$= {}^{20}C_2 = \frac{20\times19}{2\times1} = 190$

Example 2: Evaluate $^{15}C_3 + {}^{15}C_{13}$

Solution:

$^{15}C_3 + {}^{15}C_{13} = {}^{15}C_3 + {}^{15}C_2$ $(\because {}^nC_r = {}^nC_{n-r})$

$= {}^{16}C_3$ $(\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_4)$

Example 3: If $^nC_{r-1} = 36, {}^nC_r = 84$ and $^nC_{r+1} = 126$ find n and r

Solution:

We have $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

$\frac{84}{36} = \frac{n-r+1}{r}$

$\Rightarrow \frac{7}{3} = \frac{n-r+1}{r} \Rightarrow 2n - 10r + 3 = 0$ ...(1)

Also $\frac{^nC_{r+1}}{^nC_r} = \frac{n-r}{r+1} \Rightarrow \frac{126}{84} = \frac{n-r}{r+1} \Rightarrow 2n - 5r - 3 = 0$ ...(2)

Solving (1) and (2) we get, r = 3, n = 9

Example 4: Find the value of $^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{47}C_4$

Solution:

We know that $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Given $^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + ({}^{47}C_3 + {}^{47}C_4)$

$\therefore {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + ({}^{48}C_3 + {}^{48}C_4)$

$= {}^{51}C_3 + {}^{50}C_3 + ({}^{49}C_3 + {}^{49}C_4) = {}^{51}C_3 + ({}^{50}C_3 + {}^{50}C_4) = {}^{51}C_3 + {}^{51}C_4 = {}^{52}C_4$

Example 5: Find r if $^{15}C_{r+3} = {}^{15}C_{2r-3}$

Solution:

$^{15}C_{r+3} = {}^{15}C_{2r-3} \Rightarrow r+3 = 2r-3$ or $r + 3 + 2r - 3 = 15$

i.e., r = 6 or 3r = 15 i.e., r = 6 or r = 5

Example 6: Prove that the product of r consecutive positive integers is divisible by r!

Solution:

Let r consecutive positive integers be (n + 1), (n + 2), (n + 3) ....(n + r)

Their product $= (n+1)(n+2)(n+3)...(n+r)$

$= \frac{1\cdot2\cdot3\cdot n(n+1)(n+2)...(n+r)}{1\cdot2\cdot3\cdot...n}$

$= \frac{(n-r)!}{n!} = \frac{(n+r)!}{n!} \times \frac{r!}{n!}$

$\frac{(n+r)!}{r!n!} \times r! = {}^{n+r}C_r \times r!$

$\Rightarrow {}^{n+r}C_r = \frac{(n+1)(n+2)(n+3)...(n+r)}{r!}$ = Product of r consecutive integers is divisible by r!

$(\because {}^{n+r}C_r$ is a + ve integer)

Example 7: (i) If $^nC_9 = {}^nC_8$ find $^nC_{17}$ (ii) If $^nC_x = {}^nC_y$ then either x = y or x + y = n

Solution:

(i) We have $^nC_9 = {}^nC_8$

$\Rightarrow \frac{n!}{(n-9)!9!} = \frac{n!}{(n-8)!8!}$

$\Rightarrow \frac{1}{9} = \frac{1}{n-8} \Rightarrow n - 8 = 9$ or $n = 17$

$\therefore \ {}^nC_{17} = {}^{17}C_{17} = 1$

(ii) ${}^nC_x = {}^nC_y \Rightarrow {}^nC_x = {}^nC_y = {}^nC_{n-y} \quad (\because {}^nC_r = {}^nC_{n-r})$

$\Rightarrow \ x = y$ or $x + y = n$

$\Rightarrow \ x - y$ or $x = n - y$

Remark: If ${}^nC_x = {}^nC_y$ and $x \neq y$ then $x + y = n$

Example 8: If ${}^nC_8 = {}^nC_2$ find ${}^nC_2$

Solution:

${}^nC_8 = {}^nC_2$ we know that ${}^nC_x = {}^nC_y \Rightarrow x + y = n$

Using this we get $8 + 2 = n$ or $n = 10$

$\therefore \ {}^nC_2 = {}^{10}C_2 = \frac{10!}{(10-2)!\,2!} = \frac{10!}{8!\,2!} = \frac{10 \times 9}{2} = 45$

Example 9: Determine n if i) ${}^{2n}C_3 : {}^nC_3 = 12 : 1$ ii) ${}^{2n}C_3 : {}^nC_3 = 11 : 1$

Solution:

(i) Given ${}^{2n}C_3 : {}^nC_3 = 12 : 1$

$\Rightarrow \frac{{}^{2n}C_3}{{}^nC_3} = \frac{12}{1}$

$\Rightarrow \ {}^{2n}C_3 = 12\ {}^nC_2$

$\Rightarrow \frac{(2n)!}{(2n-3)!\,3!} = 12 \cdot \frac{n!}{(n-2)!\,2!}$

$\Rightarrow \frac{(2n)(2n-1)(2n-2)}{6} = 12 \cdot \frac{n(n-1)}{2}$

$\Rightarrow \ (2n-1) = 9 \Rightarrow n = 5$

(ii) We have $\frac{{}^{2n}C_3}{{}^nC_3} = \frac{11}{1} \Rightarrow \frac{(2n)!}{(2n-3)!} = \frac{11n!}{(n-3)!\,3!}$

$\Rightarrow \ (2n)(2n-1)(2n-2) = \frac{11 \cdot n(n-1)(n-2)}{6}$

$\Rightarrow \ 12(2n-1) = 33(n-2) \ \Rightarrow \ 24n - 12 = 33n - 66 \Rightarrow n = 6$

Example 10: How many chords can be drawn through 21 points on a circle.

Solution:

A chord can be drawn by joining any two points on a circle. Therefore the number of chords that can be drawn out of 21 points is ${}^{21}C_2 = \frac{21!}{19!\,2!} = 210$

Example 11: Find the number of straight lines that can be formed by joining 20 points in a plane of which 5 points are collinear. How many triangles are possible.

Solution:

$\therefore$ Number of lines from 20 points $= {}^{20}C_2 = \frac{20!}{18!\,2!} = \frac{20 \times 19}{2} = 190$

But 5 points are collinear. Therefore number of lines get reduced by ${}^5C_2 = 10$

However by joining 5 collinear points we get one line.

$\therefore$ Required number of lines $= 190 - 10 + 1 = 181$

By joining 3 non-collinear points one triangle is formed. Number of triangles from 20 non-collinear points $= {}^{20}C_3 = 1140$. But 5 points are collinear. Therefore the number of triangles reduced by ${}^5C_3 = 10$

$\therefore$ Required number of triangles $= 1140 - 10 = 1130$.

Example 12: In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls

Solution:

3 boys can be chosen from 5 boys in ${}^5C_3$ ways

3 girls can be selected from 4 girls in ${}^4C_3$ ways.

By fundamental principle the team can be selected in ${}^5C_3 \times {}^4C_3$ ways $= 10 \times 4 = 40$ ways.

Example 13: Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

Solution:

3 red balls can be selected out of 6 red balls in ${}^6C_3$ ways. Similarly 3 white balls can be selected out of 5 white balls in ${}^5C_3$ ways and 3 blue balls can be selected out of 5 blue balls in ${}^5C_3$ ways.

By Fundamental principle the number of ways in which 9 balls can be selected in

${}^{6}C_3 \times {}^{5}C_3 \times {}^{5}C_3$ ways

$\frac{6!}{3!3!} \times \frac{5!}{2!3!} \times \frac{5!}{2!3!} = 20 \times 10 \times 10 = 2000$ ways

Example 14: A man has 7 relatives, 4 of them are ladies and 3 gentlemen, his wife has 7 relatives and 3 of them are ladies and 4 gentlemen. In how many ways can they invite to a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of wife's relatives.

Solutions:

There are four possibilities

1. 3 ladies from man's side and 3 gentlemen from wife's side
   Number of ways = ${}^{4}C_3 \times {}^{4}C_3 = 4 \times 4 = 16$
2. 3 gentlemen from man's side and 3 ladies from wife's side
   Number of ways = ${}^{3}C_3 \times {}^{3}C_3 = 1 \times 1 = 1$
3. 2 ladies and 1 gentlemen from man's side and 1 lady and 2 gentlemen from wife's side
   i.e., number of ways $= ({}^{4}C_2 \times {}^{3}C_1) \times ({}^{3}C_1 \times {}^{4}C_2)$
   $= (6 \times 3) \times (3 \times 6) = 18 \times 18 = 324$
4. 1 lady and 2 gentlemen from man's side and 2 ladies and 1 gentlemen from wife's side
   Number of ways $= ({}^{4}C_1 \times {}^{3}C_2) \times ({}^{3}C_2 \times {}^{4}C_1)$
   $= (4 \times 3) \times (3 \times 4) = 144$

Hence total numbers of ways = 16 + 1 + 324 + 144 = 485 ways.

Example 15: What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit

(ii) four cards belong to four different suits

(iii) are face cards

(iv) two are red cards and two are black cards

(v) cards are of the same colour

Solution:

Number of ways of choosing 4 cards from a pack of 52 playing cards is ${}^{52}C_4 = \frac{52}{48!4!} = 270725$

(i) There are four suits namely, Heart, diamond club and spade, each having 13 cards

$\therefore$ Number of ways of choosing 4 hearts = ${}^{13}C_4$

Similarly number of ways of choosing diamonds, clubs and spades in each case is ${}^{13}C_4$

$\therefore$ Required number of ways = ${}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$

$= 4\ {}^{13}C_4$

$= 4 \times \frac{13!}{9!4!} = 2860$

(ii) There are 13 cards in each suit

Number of ways of choosing 1 card for each suit is ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards

Number of ways of choosing 4 face cards = ${}^{12}C_4 = \frac{12!}{8!\,4!} = 495$

(iv) There are 26 re cards and 26 black cards

$\therefore$ Required number of ways = ${}^{26}C_2 \times {}^{26}C_2 = 105625$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways

4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways

$\therefore$ Required number of ways = ${}^{26}C_4 + {}^{26}C_4 = 2.\ {}^{26}C_4 = 2 \cdot \frac{26!}{22!4!} = 29900$

Example 16: In how many ways can one select a cricket team of 11 from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers

Solution:

Number of players = 17

Numbers of bowlers = 5

Number of players who are not bowler = 17 – 5 = 12

Select 7 players who are not bowlers out of 12 players in ${}^{12}C_7$ ways. 4 bowlers can be selected out of 5 bowlers in ${}^{5}C_4$ ways.

$\therefore$ Total number of ways = ${}^{12}C_7 \times {}^{5}C_4 = \frac{12!}{5!7!} - \frac{5!}{1!4!} = 3960$

Example 17: In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student.

Solution:

Total number of courses available = 9

Number of courses which are not compulsory = 9 – 2 = 7

As two specific courses are compulsory, a student can choose them in ${}^2C_2$ ways. The remaining 3 courses can be chosen from 7 non-compulsory courses in ${}^7C_3$ ways.

$\therefore$ Total number of ways = ${}^7C_3 \times {}^2C_2 = \frac{7!}{4!3!} \times 1 = \frac{7\times6\times5}{3\times2\times1} = 35$

Example 18: In a examination a question paper consists 12 questions divided into parts, part I and part II containing 5 and 7 questions respectively. A student is required to attempt 8 questions, selecting atleast 3 from each part. In how many ways can a student select the questions.

Solution:

The following are the possibilities

1. 3 questions from part I and 5 questions from part II

   $= {}^7C_3 \times {}^5C_2 = 35 \times 1 = 35$

2. 4 questions from part I and 4 questions from part II

   $= {}^7C_4 \times {}^5C_4 = 35 \times 5 = 175$

3. 5 questions from part I and 3 questions from part II

   $= {}^7C_5 \times {}^5C_3 = 21 \times 10 = 210$

   Hence the required number of selections = 35 + 175 + 210 = 420

Example 19: A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of

(i) exactly 3 girls (ii) at least 3 girls (iii) at most 3 girls

Solution:

(i) Exactly 3 girls

$${}^9C_4 \times {}^4C_3 = \frac{9!}{5!4!} \times \frac{4!}{3!} = 504$$

(ii) Atleat 3 girls. The possibilities are a) 4 boys and 3 girls b) 3 boys and 4 girls.

i.e., a) ${}^9C_4 \times {}^4C_3 = 504$ and b) ${}^9C_3 \times {}^4C_4 = 84$

i.e., Hence the total number of committees = 504 + 84 = 588

(iii) **At mot 3 girls:** The following are the possibilities

a) 7 boys b) 6 boys and 1 girl c) 5 boys and 2 girls d) 4 boys and 3 girls

i.e., a) ${}^9C_1 = 36$ b) ${}^9C_6 \times {}^4C_1 = 84 \times 4 = 336$

c) ${}^9C_5 \times {}^4C_2 = 126 \times 6 = 756$ d) ${}^9C_4 \times {}^4C_3 = 504$

$\therefore$ Total number of committees 36 + 336 + 756 + 504 = 1092

Example 20: If m parallel lines in a plane are intersected by a family of a n parallel lines, find the number of parallelogram formed.

Solution:

A parallelogram is formed by selecting straight lines from the set of m parallel lines and 2 straight lines from the set of n parallel lines and 2 straight lines from the set of a parallel lines. This can be done in ${}^mC_2$ and ${}^nC_2$ ways respectively.

Hence the required number of parallelograms

$$= {}^mC_2 \times {}^nC_2 = \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}$$

Example 21: How many diagonals can be formed by joining the angular points of a polygon of (i) 12 sides (ii) n sides.

Solution:

(i) A diagonal is obtained by joining two points. Thus the number of diagonals obtained by joining any two points out of 12 is given by

${}^{12}C_2 - 12 = 66 - 12 = 54$

(ii) ${}^nC_2 - n$

Example 22: In a small village there are 87 families of which 52 families have atmost 2 children. In a rural development programme, 20 families are to be chosen for assistance, of which atleast 18 families must have atmost 2 children. In how many ways can the choice be made.

Solution:

The possibilities are

(i) 18 families with atmost 2 children and 2 from remaining 35.

(ii) 19 families with atmost 2 children and 1 from remaining 35.

(iii) 20 families with atmost 2 children only.

∴ The required number of ways = ${}^{52}C_{18} \times {}^{35}C_2 + {}^{52}C_{19} \times {}^{35}C_1 + {}^{52}C_{20}$

EXERCISE:

1. **Evaluate**

(i) ${}^{13}C_6 + {}^{13}C_5$ (ii) ${}^{31}C_{26} - {}^{30}C_{26}$

(iii) $\sum_{r=1}^{5} {}^5C_r$ (iv) ${}^{100}C_{100}$

(v) ${}^{50}C_1$ (vi) ${}^{11}C_9$

(vii) ${}^{61}C_{51} - {}^{60}C_{56}$ (viii) ${}^{10}C_4 + {}^{10}C_5$

2. (i) If ${}^nC_{10} = {}^nC_{12}$ find n and hence find ${}^nC_{22}$

(ii) If ${}^nC_7 = {}^nC_5$ find n

(iii) If ${}^nC_8 = {}^nC_6$ find ${}^nC_2$

(iv) If ${}^{18}C_r = {}^{18}C_{r+2}$ find r and hence find ${}^rC_5$

3. If ${}^nP_r = 1680$ and ${}^nC_r = 70$ find n and r

4. Prove that

(i) ${}^2C_1 + {}^3C_1 + {}^4C_1 = {}^3C_2 + {}^4C_2$

(ii) ${}^3C_3 + {}^3C_1 + {}^4C_2 = {}^5C_3$

(iii) $2 \times {}^7C_4 = {}^8C_4$

(iv) ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 = 0$

5. *(i) If ${}^{2n}C_3 : {}^nC_2 = 44 : 3$ find n*

(ii) If ${}^{16}C_r = {}^{16}C_{r+2}$ find ${}^rC_4$

6. ***If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P. find n.***

7. ***Prove that***

(i) $n \cdot {}^{n-1}C_{r-1} = {}^{(n-r+1)n}C_{r-1}$ (ii) ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$

(iii) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{(n-r+1)}{r}$ (iv) ${}^nC_{r+1} + {}^nC_{r-1} + 2 \cdot {}^nC_r = {}^{n+2}C_{r+1}$

8. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$ find n and r

9. (i) If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$ find n

(ii) If ${}^nC_r : {}^nC_{r+1} = 1 : 2$ and ${}^nC_{r+1} : {}^nC_{r+2} = 2 : 3$ find n and r

10. Find the number of straight lines that can be drawn out of 10 points of which 7 are collinear.

11. Find the number of diagonals that can be drawn by joining the vartices of an octagon.

12. There are 15 points in a plane, no three of which are in a straight line, except 6, all of which are in straight line. Find the number of straight lines which can be drawn by joining them.

13. From a class 32 student, 4 are to be chosen for a sports completion. In how many ways can this be done.

14. A committee of 2 girls is to be selected from 4 girls. In how many ways can this be done.

15. Fifteen persons meet in a room and each shakes hand with all the others. Find the number of hand shakes.

16. Sushanth wants to choose any 9 stamps from a set of 11 different stamps. How many different selections can be make.

17. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 course are compulsory for every student.

18. In how many ways can a cricket team of eleven be chosen out of 15 players if

(i) there is no restriction on the selection

(ii) a particular player is always chosen

(iii) a particular player is never chosen

19. A committee of 12 is to be formed from 9 girls and 8 boys. In how many ways this can be done if atleast 5 ladies have to be included in a committee? In how many of these committees

(i) The girls are in majority

(ii) the boys are in majority

20. A polygon has 44 diagonals. Find the number of its sides.

21. A student has to answer 10 questions, choosing at least 4 from each of Part A and Part B. If there are 6 questions in Part A and 7 in Part B, in how many ways can the student choose 10 questions.

22. Out of 18 points in a plane, no three are in the same straight line except 5 points which are collinear. How many (i) straight lines (ii) triangles can be formed by joining them.
23. A committee of 3 persons is to be constituted from a group of 2 man and 3 women. In how may ways can this be done? How many of these committees would consists of 1 mm and 2 women.
24. A bag contains 4 red, 3 white and 2 blue balls. Three balls are drawn at random out of the bag. Determine the number of ways of selecting at atleast one white balls in the selection.
25. A student is allowed to select at most a n books from a collection of (2n + 1) books. If the number of ways in which he can do this is 64, find n.
26. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen.
27. In how many ways can a football team of 11 players be selected from 16 players. How many of them will (i) include 2 particular players (ii) exclude 2 particular players.
28. Find the number of ways in which 5 cards can be selected out of deck of 52 cards if atleast one of the 5 cards is an ace.

ANSWERS:

1. (i) 3003 (ii) 142506 (iii) 31 (iv) 1
   (v) 50 (vi) 55 (vii)34220 (viii) 462
2. (i) 22, 1 (ii) 12 (iii) 91 (iv) 8, 56
3. $n = 8, r = 4$
5. (i) 6 (ii) 35
6. 14, 7
8. $n = 3, r = 2$
9. (i) $n = 19$ (ii) $n = 14, r = 4$
10. 25 11. 20 12. ${}^{15}C_2 - {}^{6}C_2 + 1$
13. ${}^{32}C_4$ 14. 6 15. 105
16. 55 17. 35

18. (i) 1365 (ii) 1001 (iii) 364
19. 6062, 2702, 1008 20. 11 21. 266
22. 144, 806 23. 10, 6 24. 64
25. 3 26. 817190 27. ${}^{14}C_9$, ${}^{14}C_{11}$ 28. 886656

2.8 Binomial coefficients:

Introduction: An expression consisting of two terms is called a binomial expression.

For example: (x + a), (a + b), (x + y), (7x – 4y) etc The process of finding higher powers of binomials such as $(x + a)^{10}$, $(x + a)^{100}$ etc becomes more difficult, and hence look for a general formula which will help us in finding higher powers of a binomial. The term **'binomial coefficients'** was first introduced by the German mathematician **Michel stipel (1486 – 1567).** The ancient mathematicians knew about this binomial coefficients in the expansion $(x + a)^n$ $(0 \le n \le 7)$ and the arrangement of these coefficients was in the form of a diagram called **'Meru prastara'** provided by **Pingala.** (the creator of binary system)

Binomial theorem for positive integers:

Statement: If 'x' and 'a' are real numbers and 'n' is any positive integer then $(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + {}^nC_n a^n$

We prove the above theorem by using the principle of mathematical induction.

Note:

(i) The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots {}^nC_n$ which are also denoted by $C_0, C_1, C_2, \ldots, C_r, \ldots C_n$ are called binomial coefficients. They are also denoted by $\binom{n}{0}, \binom{n}{1}, \binom{n}{2} \ldots \binom{n}{r} \ldots \binom{n}{n}$

(ii) The general term in the binomial expansion is (r + 1)th term and is given by

$T_{r+1} = {}^nC_r x^{n-r} a^r$

Where 'n' is power of the binomial, x is the first term and 'a' the second term of the binomial.

(iii) The number of terms in any binomial expansion is always one more than the power of the binomial.

Special cases of Binomial Expansion:

(i) changing a to –a in the binomial theorem we have,

$$(x-a)^n = (x+(-a))^n$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}(-a) + \binom{n}{2}x^{n-2}(-a)^2 + \dots + \binom{n}{r}x^{n-r}(-a)^n$$

$$= \binom{n}{0}x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + \dots + (-1)^n\binom{n}{n}a^n$$

$$(x-a)^n = C_0x^n - C_1x^{n-1}a + C_2x^{n-2}a^2 - C_3x^{n-3}a^3 + \dots + (-1)^n C_n a^n$$

We observe that RHS are alternately positive and negative and sign of the last term depends on whether 'n' is even or odd.

(ii) Prove that for a positive integer 'n' $C_0 + C_1 + C_2 + \dots + C_n = 2^n$

Solution:

We have

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$$

Replacing x by 1 and a by 1, we get

$$(1+1)^n = \sum_{r=0}^{n} {}^nC_r 1^{n-r} 1^r = \sum_{r=0}^{n} {}^nC_r$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

$$2^n = C_0 + C_1 + C_2 + \dots + C_n$$

Hence the result.

(iii) Prove that for a positive integer 'n'

$$C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0$$

Solution:

We have

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$$

Taking x = 1 and a = –1 we get

$$(1-1)^n = \sum_{r=0}^{n} {}^nC_r 1^{n-r}(-1)^r$$

$$0 = \sum_{r=0}^{n} {}^nC_r(-1)^r = \sum_{r=0}^{n}\binom{n}{r}(-1)^r = \sum_{r=0}^{n} C_r(-1)^r$$

$$0 = C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n$$

Hence the result.

WORKED EXAMPLES:

Example 1: Find the coefficient of x^5 in $(x+3)^8$.

Solution:

We have, $T_{r+1} = {}^nC_r x^{n-r} a^r = {}^8C_r x^{8-r} 3^r$

Comparing the indices of x in x^5 and in T_{r+1} we get $8 - r = 5 \Rightarrow r = 3$

$\therefore T_4 = {}^8C_3 x^{8-3} 3^3 = {}^8C_3 x^5 27$

Hence the coefficient of x^5 is ${}^8C_3 \cdot 27$

Example 2: Find the coefficient of $\frac{1}{x^{17}}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Solution:

We have, $T_{r+1} = {}^nC_r x^{n-r} a^r$

i.e., $T_{r+1} = {}^{15}C_r (x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r$

$$= {}^{15}C_r x^{60-4r}\left(-\frac{1}{x^3}\right)^r = {}^{15}C_r x^{60-4r-3r}(-1)^r$$

$$= {}^{15}C_r x^{60-7r}(-1)^r$$

Comparing the indices of x in x^{-17} and in T_{r+1}; we have $-17 = 60 - 7r \Rightarrow 7r = 77 \Rightarrow r = 11$

$\therefore T_{12} = {}^{15}C_{11} x^{60-77}(-1)^{11} = -({}^{15}C_{11})x^{-17}$

$$= \frac{-({}^{15}C_{11})}{x^{17}}$$

Hence the coefficient of $\frac{1}{x^{17}}$ is $-({}^{15}C_{11})$

Example 3: Find the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$

Solution:

We have $T_{r+1}=\binom{n}{r}x^{n-r}a^r$

i.e., $T_{r+1}=\binom{12}{r}(2x)^{12-r}(-3y)^r$

$={}^{12}C_r2^{12-r}x^{12-r}(-3)^ry^r$

$={}^{12}C_r2^{12-r}(-3)^rx^{12-r}y^r$

By taking r = 3, we have $T_4={}^{12}C_32^9(-3)^3x^9y^3$

Example 4: Find the coefficient of x^6y^3 in the expansion of $(x+2y)^9$

Solution:

We have, $T_{r+1}={}^nC_rx^{n-r}a^r$

i.e., $T_{r+1}={}^9C_r(x)^{9-r}(2y)^r$

$={}^9C_r\,2^rx^{9-r}y^r$

By taking r = 3 we have

$T_4={}^9C_32^3x^6y^3$

$\therefore$ The coefficient of x^6y^3 is ${}^9C_3\times2^3=672$

Example 5: Find the term independent of x in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^{15}$ $(x\neq0)$

Solution:

We have,

$T_{r+1}={}^nC_r\,x^{n-r}a^r={}^{15}C_r\left(\frac{3x^2}{2}\right)^{15-r}\left(-\frac{1}{3x}\right)^r=(-1)^r\,{}^{15}C_r\left(\frac{3}{2}\right)^{15-r}x^{30-2r-r}\left(\frac{1}{3}\right)^r$

$=(-1)^r\,{}^{15}C_r\left(\frac{3}{2}\right)^{15-r}x^{30-3r}\left(\frac{1}{3}\right)^r$

The term independent of x means, index of x is zero. Comparing the indices of x in x^0 and in *T_{r+1}, we get $0=30-3r\Rightarrow r=10$*

$\therefore\ T_{11}=(-1)^{10}\,{}^{15}C_{10}\left(\frac{3}{2}\right)^5x^{30-30}\times\left(\frac{1}{3}\right)^{10}$

$={}^{15}C_{10}\frac{3^5}{2^5}\times\frac{1}{3^{10}}=\frac{{}^{15}C_{10}}{2^5\cdot3^5}=\frac{{}^{15}C_{10}}{(6)^5}$ which is independent of x.

Example 6: Find the term independent of x in the expansion of $\left(x-\frac{3}{x^2}\right)^{18}$ $(x\neq0)$

Solution:

We have, $T_{r+1}={}^nC_rx^{n-r}a^r={}^{18}C_r\,x^{18-r}\left(-\frac{3}{x^2}\right)^r$

$={}^{18}C_r\,x^{18-r}\frac{(-1)^r3^r}{x^{2r}}={}^{18}C_r\,x^{18-r-2r}\,3^r(-1)^r$

$=(-1)^r\,{}^{18}C_r\,x^{18-3r}\,3^r$

Comparing x^{18-3r} with x^0 we get $18-3r=0\Rightarrow r=6$

$\therefore\ T_4=(-1)^3\,{}^{18}C_3x^{18-18}\cdot3^6$

$T_4=-\,{}^{18}C_3\cdot3^6$ which is independent of x

Example 7: If the coefficients of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, find n.

Solution:

We have, $T_{r+1}={}^nC_r\,x^{n-r}a^r$

i.e., $T_{r+1}={}^nC_r(2)^{n-r}\left(\frac{x}{3}\right)^r$

Now $x^r=x^7\Rightarrow r=7$

and $x^r=x^8\Rightarrow r=8$

$\therefore\ \ T_8={}^nC_7(2)^{n-7}\left(\frac{x}{3}\right)^7$

$$T_9 = {}^nC_8(2)^{n-8}\left(\frac{x}{3}\right)^8$$

Given ${}^nC_7(2)^{n-7}\left(\frac{x}{3}\right)^7 = {}^nC_8(2)^{n-8}\left(\frac{x}{3}\right)^8$

$$\Rightarrow \frac{n!}{(n-7)!\,7!}\frac{2^{n-7}}{3^7} = \frac{n!}{(n-8)!\,8!}2^{n-8}\frac{1}{3^8}$$

$$\Rightarrow \frac{2^{n-7}}{(n-7)} = \frac{2^{n-8}}{8\times 3} \Rightarrow \left(2^{n-7-n+8}\right)24 = n-7$$

$$\Rightarrow 48 = n-7 \Rightarrow n = 55$$

Example 8: Find the coefficient of x^4 in the product $(1 + 2x)^4 (2 - x)^5$ using binomial theorem.

Solution:

$$(1+2x)^4 = {}^4C_0 + {}^4C_1(2x) + {}^4C_2(2x)^2 + {}^4C_3(2x)^3 + {}^4C_4(2x)^4$$
$$= 1 + 8x + 6(4x^2) + 4(8x^3) + 16x^4$$
$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$$
$$(2-x)^5 = {}^5C_0 2^5 - {}^5C_1 2^4 x + {}^5C_2 2^3 x^2 - {}^5C_3\, 2^2 x^3 - {}^5C_4 2x^4 + {}^5C_5\, x^5$$
$$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$
$$\therefore (1+2x)^4(2-x)^5 = (1+8x+24x^2+32x^3+16x^4)(32-80x+80x^2-40x^3+10x^4-x^5)$$
$$= 1(10x^4) + 8x(-40x^3) + 24x^2(80x^2) + 32x^3(-80x) + 16x^4(32)$$
$$= 10x^4 - 320x^4 + 1920x^4 - 2560x^4 + 512x^4$$
$$= -438x^4 \quad \therefore \text{Coefficient of } x^4 \text{ is } -438$$

Example 9: Find the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Solution:

$$(1+x+x^2+x^3)^{11} = \left[(1+x)+x^2(1+x)\right]^{11}$$
$$= \left[(1+x)(1+x)^2\right]^{11}$$
$$= (1+x)^{11}(1+x^2)^{11}$$

$$(1+x)^{11}(1+x^2)^{11} = \left({}^{11}C_0 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + \ldots\right)$$
$$\left({}^{11}C_0 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + {}^{11}C_3 x^6 + \ldots\right)$$
$$= \left(1 + 11x + 56x^2 + 165x^3 + 330x^4\right)\left(1 + 11x^2 + 55x^4\right) \text{ other terms are deleted}$$

Coefficient of $x^4 = 55 + 605 + 330 = 990$

Generalized Binomial theorem: (or Multi-nomial theorem)

If 'n' and 'r' are positive integers, the coefficients of $x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$ in the expansion of $(x_1 + x_2 + x_3 + \ldots + x_r)^n$ is $\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_r!}$ and is denoted by $\binom{n}{n_1, n_2, \ldots, n_r}$.

(where each n_i is a non-negative integer $\le n$ and $n_1 + n_2 + n_3 + \ldots + n_r = n$)

WORKED EXAMPLES:

Example 1: Evaluate the following

(i) $\binom{6}{1, 2, 3}$ **(ii)** $\binom{9}{3, 3, 3, 0}$ **(iii)** $\binom{12}{5, 3, 3, 2}$

Solution:

(i) $\frac{6!}{1!\,2!\,3!} = \frac{720}{12} = 60$

(ii) $\frac{9!}{3!\,3!\,3!\,0!} = \frac{362880}{216} = 1680$

(iii) $\frac{12!}{5!\,3!\,3!\,0!} = 166320$

Example 2: Find the coefficient of $x^2 y^2 z^3$ in the expansion of $(x + y + z)^7$.

Solution:

The general term is given by $\binom{n}{n_1, n_2, n_3} x^{n_1} y^{n_2} z^{n_3}$

where $(n_1, n_2, n_3) = (2, 2, 3)$ and $n = 7$

$\therefore$ We have $\binom{7}{2, 2, 3} = \frac{7!}{2!\,2!\,3!} = \frac{5040}{24} = 210$

Example 3: Determine the coefficient of $x^2y^2z^3$ in the expansion of $(3x - 2y - 4z)^7$

Solution:

The general term is given by $\binom{n}{n_1, n_2, n_3}(3x)^{n_1}(-2y)^{n_2}(-4z)^{n_3}$

By taking $n_1 = 2$, $n_2 = 2$ and $n_3 = 3$ this becomes $\binom{7}{2, 2, 3}(3x)^2(-2y)^2(-4z)^3$

$$\binom{7}{2, 2, 3}3^2(-2)^2(-4)^3 x^2y^2z^3$$

The required coefficient is

$$-(9\times4\times64)\frac{7!}{2!\,2!\,3!} = -483840$$

Example 4: Determine the coefficient of xyz^2 in the expansion of $(2x - y - z)^4$

Solution:

The general term is given by, $\binom{4}{n_1, n_2, n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$

By taking $n_1 = 1$, $n_2 = 1$, $n_3 = 2$, we have

$$\binom{4}{1, 1, 2}\left((2x)^1(-y)^1(-z)^2\right) = \frac{4!}{1!\,1!\,2!}(-2xyz^2) = -24\ xyz^2$$

$\therefore$ The required coefficient $= -24$

Example 5: Find the coefficient of $a^{11}b^4$ in the expansion of $(2a^3 - 3ab^2 + c^2)^6$

Solution:

The general term is given by,

$$\binom{6}{n_1, n_2, n_3}(2a^3)^{n_1}(-3ab^2)^{n_2}(c^2)^{n_3}$$

i.e., $\binom{6}{n_1, n_2, n_3}2^{n_1}(-3)^{n_2}a^{3n_1+n_2}b^{2n_2}c^{2n_3}$ ...(1)

We must have $3n_1 + n_2 = 11$, $2n_2 = 4$ which gives $n_2 = 2$ and $n_1 = 3$

Further $n_1 + n_2 + n_3 = 6 \Rightarrow n_3 = 1$ $\therefore$ (1) becomes $\binom{6}{3, 2, 1}2^3(-3)^2a^{11}b^4c^2$.

The coefficient of $a^{11}b^4$ is $\frac{6!}{3!\,2!\,1!}\times8\times9\times c^2 = 4320\ c^2$

Hence the required coefficient is $4320\ c^2$

EXERCISE:

1. Find the coefficient of x^7 in the expansion of $\left(x - \frac{1}{x^2}\right)^{40}$
2. Find the coefficient of x^{11} in the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{12}$
3. Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$
4. Find the coefficient of x^4 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^8$
5. Find the coefficient of x^5y^2 in the expansion of $(2x - 3y)^7$.
6. Find the coefficient of x^9y^3 in the expansion of $(x + 2y)^{12}$
7. Find the coefficient of x^5 in the expansion of $(1 + 2x)^6(1 - x)^7$
8. Find the coefficient of x^{10} in the expansion of $(1 + x + x^2)(1 - x)^{15}$
9. Show that the coefficients of x^p and x^q in the expansion of $(1 + x)^{p+q}$ are equal.
10. In the expansion of $(2 + x)^{50}$, 17^{th} and 18^{th} terms are equal, show that $x = 1$
11. If the coefficient of $(2r + 4)^{th}$ and $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal, show that $r = 6$.
12. Find the term independent of x in the expansion of
    (i) $\left(\frac{4x^2}{3} + \frac{3}{2x}\right)^9$ (ii) $\left(x - \frac{3}{x^2}\right)^{18}$
13. Evaluate: (i) $\binom{7}{2, 3, 2}$ (ii) $\binom{10}{5, 3, 3, 2}$
14. Find the coefficient of $x^3y^3z^2$ in the expansion of $(2x - 3y + 52)^8$
15. Find the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$

ANSWERS:

(1) $-{}^{40}C_{11}$ (2) -25344 (3) 672 (4) ${}^8C_4 2^4 3^4$ (5) 6048

(6) 1760 (7) 171 (8) 4433 (12) (i) ${}^9C_3 \times 27$ (ii) ${}^{18}C_6 3^6$

(13) (i) 210 (ii) Meaning less (14) -3024000 (15) $\left(\frac{125}{6}\right)16!$

2.9 Recurrence Relations:

A **'recurrence relation'** is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s).

The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. If we denote the nth term in the sequence by a_n, such a recurrence relation is of the form $a_{n+1} = f(a_n)$ for some function of one such example is $a_{n+1} = 2 - \frac{a_n}{2}$.

A recurrence relation can also be of higher order, where the term a_{n+1} could depend not only on the previous term a_n, but also on earlier terms such as a_{n-1}, a_{n-2} etc.

A second order recurrence relation depends just on a_n and a_{n-1} and is of the form $a_{n+1} = f(a_n, a_{n-1})$

for some function f with two inputs. For example, the recurrence relation $a_{n+1} = a_n + a_{n-1}$ can generate the Fibonacci numbers.

To generate sequence based on a recurrence relation, one must start with some initial values. For a first order recursion $a_{n+1} = f(a_n)$, one just needs to start with an initial value a_0 and can generate all remaining terms using the recurrence relation. For a second order recursion $a_{n+1} = f(a_n, a_{n-1})$ one needs to begin with two values a_0 and a_1. Higher order recurrence relations require correspondingly more initial values.

2.10 First order linear recurrence relation:

A first order linear recurrence relation with constant coefficients is of the form

$a_n = ka_{n-1} + f(n), (n \geq 1)$ ...(1)

where k is a known constant and f(n) is a known function. If f(n) = 0, then the recurrence relation is said to be **'homogeneous'**, otherwise **'non-homogeneous'**.

Method of finding general solution of (1):

When $n = 1$, $a_1 = ka_0 + f(1)$

When $n = 2$, $a_2 = ka_1 + f(2)$

i.e., $a_2 = k[ka_0 + f(1)] + f(2)$

$a_2 = k^2 a_0 + kf(1) + f(2)$

When n = 3, $a_3 = ka_2 + f(3)$

i.e., $a_3 = k[k^2 a_0 kf(1) + f(2)] + f(3)$

$a_3 = k^3 a_0 + k^2 f(1) + kf(2) + f(3)$

Proceeding like this, we have

$a_n = k^n a_0 + k^{n-1} f(1) + k^{n-2} f(2) + k^{n-3} f(3) + \ldots + kf(n-1) + f(n)$

or $a_n = k^n a_0 + \sum_{r=1}^{n} k^{n-r} f(r)\ n \geq 1$ ...(2)

which is the general solution of recurrence relation (1).

Suppose f(n) = 0, then the general solution of the homogeneous linear recurrence relation $a_n = ka_{n-1}$ is given by $a_n = k^n a_0 (n \geq 1)$ ...(3)

Note: If a_0 is specified then the solutions of (2) and (3) gives particular solution. The specified value of a_0 is called **initial condition**.

WORKED EXAMPLES:

Example 1: Solve the recurrence relation $a_{n+1} = 4\, a_n$ for $n \geq 0$ given $a_0 = 3$

Solution:

The given relation is homogeneous.

∴ General solution is $a_n = 4^n a_0$ for $n \geq 1$

Put $a_0 = 3$, we get $a_n = 3(4^n)$ for $n \geq 1$

which is the particular solution.

Example 2: Solve the recurrence relation $3a_{n+1} - 4a_n = 0, n \geq 0, a_1 = 5$

Solution:

We have $3a_{n+1} - 4a_n = 0$

$\Rightarrow 3a_{n+1} = 4\, a_n$

or $a_{n+1} = \frac{4}{3} a_n\ n \geq 0$

which is homogeneous and hence general solution is $a_n = \left(\frac{4}{3}\right)^n a_0$ for $n \geq 1$

for $n = 1$, $a_1 = \frac{4}{3} \cdot a_0$ given $a_1 = 5$

$\therefore\ 5 = \frac{4}{3}a_0$ or $a_0 = \frac{15}{4}$

Hence particular solution is $a_n = \left(\frac{4}{3}\right)^n \cdot \frac{15}{4} (n \geq 1)$

Example 3: Solve the recurrence relation $a_n = 7a_{n-1}$ where $n \geq 1$ gives that $a_2 = 98$

Solution:

The given recurrence relation is homogeneous.

we have $a_n = 7a_{n-1}$, we can re-write this as $a_{n+1} = 7a_n$ for $n \geq 0$. Hence the general solution is $a_n = 7^n a_0$ for $n \geq 1$.

For n = 2, $a_2 = 7^2 a_0$ given $a_2 = 98$

$\therefore\ 98 = 49\ a_0 \Rightarrow a_0 = 2$

Hence the particular solution is $a_n = 7^n . 2\ (n \geq 1)$

Example 4: Obtain the solution of the recurrence relation $a_n = na_{n-1}$ for $n \geq 1$ given that $a_0 = 1$

Solution:

We have, $a_n = na_{n-1}$

For n = 1, $a_1 = a_0 = 1!\ a_0$

For n = 2, $a_2 = 2a_1 = 2a_0 = (2 \times 1)a_0 = 2!\ a_0$

For n = 3, $a_3 = 3a_2 = 3(2a_0) = (3 \times 2 \times 1)\ a_0 = 3!\ a_0$

Similarly $a_4 = (4 \times 3 \times 2 \times 1)a_0 = 4!a_0$

....

$a_n = n!\ a_0$ for $n \geq 1$

Given $a_0 = 1$, $\therefore\ a_n = n!$ which is the required solution

Example 5: Obtain the solution of the recurrence relation $a_{n+1} = 2a_n + 5, n \geq 0, a_0 = 1$

Solution:

We have $a_{n+1} = 2a_n + 5$

or $a_n = 2a_{n-1} + 5$ which is a non-homogeneous recurrence relation of the form $a_n = ka_{n-1} + f(n)$

where k = 2 and f(n) = 5

Hence the required solution is given by

$$a_n = k^n a_0 + \sum_{r=1}^{n} k^{n-r} f(r) \qquad (n \geq 1)$$

i.e., $$a_n = 2^n a_0 + \sum_{r=1}^{n} 2^{n-r}(5)$$

$$= 2^n a_0 + 5\left(2^{n-1} + 2^{n-2} + ... + 2^2 + 2^1 + 1\right)$$

$$= 2^n a_0 + 5\left(\frac{2^n - 1}{2 - 1}\right) (\because \text{ they are in G.P.})$$

$$= 2^n a_0 + 5\left(2^n\right) - 5$$

$$a_n = (a_0 + 5)2^n - 5 \text{ is the general solution}$$

Given $a_0 = 1$

$\therefore\ a_n = 6 \cdot 2^n - 5$ is the particular solution

Example 6: Obtain the solution of the recurrence relation $a_{n+1} = a_n + (2n + 3)\ n \geq 0, a_0 = 1$

Solution:

We have $a_{n+1} = a_n\ (2n + 3), n \geq 0, a_0 = 1$ changing n to (n – 1), we get

$a_n = a_{n-1} + (2n + 1)$

Which is non-homogeneous recurrence relation with k = 1 and f(n) = (2n + 1)

Hence general solution is given by

$$a_n = k^n a_0 + \sum_{r=1}^{n} k^{n-r} f(r)$$

$$= 1 \cdot a_0 + \sum_{r=1}^{n} 1(2r + 1)$$

$$a_n = a_0 + \{3 + 5 + 7 + ... + (2n + 1)\}$$

The numbers 3, 5, 7,... are in A.P. and hence their sum is given by $\frac{n}{2}\{2a + (n - 1)d\}$

where a = 3 and d = 2

i.e., $$a_n = a_0 + \left[\frac{n}{2}\{2(3) + (n - 1)2\}\right]$$

$$= a_0 + \left[n\{3 + n - 1\}\right] = a_0 + \left[2n + n^2\right]$$

$$a_n = a_0 + n^2 + 2n$$

which is the general solution

Given $a_0 = 1$

$\therefore a_n = 1 + n^2 + 2n = (n+1)^2$ is the particular solution.

Example 7: Solve $a_n - a_{n-1} = 1 + 3n + 3n^2$ ($n \geq 1$), $a_0 = 1$

Solution:

We have, $a_n = a_{n-1} + 1 + 3n + 3n^2, n \geq 1$

which is a non-homogeneous recurrence relation and hence the general solution is of the form

$$a_n = k^n a_0 + \sum_{r=1}^{n} k^{n-r} f(r)$$

where $k = 1$ and $f(r) = 1 + 3r + 3r^2$

$$\therefore a_n = 1 \cdot a_0 + \sum_{r=1}^{n} 1 \cdot (1 + 3r + 3r^2) = a_0 + \sum_{r=1}^{n} 1 + 3\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} r^2$$

$$= a_0 + n + \frac{3n(n+1)}{2} + 3\frac{n(n+1)(2n+1)}{6}$$

(Since we know that $\sum n = \frac{n(n+1)}{2}$ and $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$)

$$a_n = a_0 + n + \frac{3}{2}n(n+1) + \frac{1}{2}n(n+1)(2n+1)$$

Hence the general solution is of the form $a_n = k^n a_0 + \sum_{r=1}^{n} k^{n-r} f(r)$

i.e., $a_n = a_0 + \frac{n}{2}\{2 + 3n + 3 + (2n^2 + 3n + 1)\}$

$$= a_0 + \frac{n}{2}\{6 + 6n + 2n^2\}$$

or $a_n = a_0 + n(n^2 + 3n + 3)$ which is the general solution.

By putting $a_0 = 1$ (given)

We get $a_n = 1 + 3n + 3n^2 + n^3 = (n+1)^3$

Which is the particular solution

Example 8: Find the recurrence relation and the initial condition for the sequence 2, 10, 50, 250, ... Hence find the general solution of the sequence.

Solution:

From the given sequence, note that $a_0 = 2, a_1 = 10, a_2 = 50, a_3 = 250, \ldots$

i.e., $a_1 = 5a_0, a_2 = 5a_1, a_3 = 5a_2 \ldots$

Therefore we write $a_n = 5a_{n-1}$ for $n \geq 1$ with $a_0 = 2$ which is the required recurrence relation.

The general solution is given by $a_n = k^n a_0$ i.e., $a_n = 5^n a_0$

Example 9: Find a recurrence relation and initial condition for the sequence 1, 5, 9, 13, ... Hence find the general solution of the sequence

Solution:

The given sequence is 1, 5, 9, 13, ... Here $a_0 = 1, a_1 = 5, a_2 = 9, a_3 = 13, \ldots$

Evidently $a_1 = 4 + a_0, a_2 = 4 + a_1, a_3 = 4 + a_2, \ldots$

$\therefore a_n = 4 + a_{n-1}$ ($n \geq 1$)

or $a_n = a_{n-1} + 4$ with $a_0 = 1$

which is the recurrence relation for the given sequence, for $n \geq 1$ with $a_0 = 1$ as the initial condition.

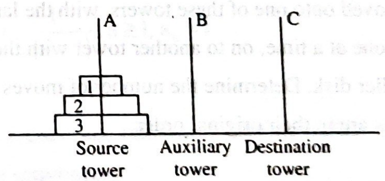
The recurrence relation is non-homogeneous where $k = 1$ and $f(n) = 4$, $a_0 = 1$

$\therefore$ General solution $a_n = k^n a_0 + \sum_{r=1}^{n} k^{n-r} f(r)$ i.e., $a_n = a_0 + \sum_{r=1}^{n} 4 = a_0 + 4\sum_{r=1}^{n} 1 = a_0 + 4n$

$a_0 = 1$ and $\therefore a_n = 1 + 4n$ is the general solution.

2.11 The Towers of Honoi problem:

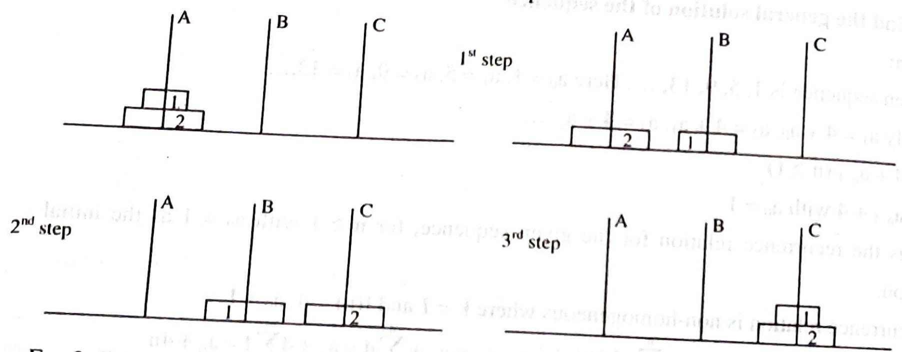
'Towers of Hanoi' is a classical problem that can be solved using the concept of recursion relation. In this mathematical puzzle we have 3 towers and n disks. The objective of the problem is to move the entire disks to another tower obeying the following rules.



A is the source tower, where we have disks. All the disks, we have to move finally to destination tower C, through the tower B, which helps in moving the disks and hence is called the Auxiliary tower.

Rules:

1. Only one disk can be moved among the towers at any given time.
2. Only the top disk can be removed i.e., each move consists of taking the upper disk from one of the poles and placing it on the top of another pole.
3. No disk may be placed on top of a smaller disk. (i.e., No large disk can sit on a small disk) upper most disk should be taken to the next pole.



For 2 disks to shift to the destination C, 3 steps is necessary. Only three not less than or not more than that. Similarly to move 3 disks to the destination C, 7 steps is a must.

i.e., for 2 disks to move, number of steps 3 i.e., $2^2 - 1 = 3$. For 4 disks to move, number of steps 15. i.e., $2^4 - 1$. Continuing this process for n disks to move number of steps $2^n - 1$.

Statement of the Towers of Hanoi problem:

There are 3 towers fixed vertically on a table and 'n' circular disks having holes at the centres and having increasing diameters are moved onto one of these towers, with the largest disc at the bottom. The disks are to be transfered one at a time, on to another tower with the condition that at no time a larger disk is put on a smaller disk. Determine the number of moves for the transfer of all the n disks, so that at end the disks are in their original order.

Solution:

Let a_n be the number of moves required to transfer 'n' disks. Evidently a_0 has no meaning, therefore we say $a_0 = 0$. If you have one disk, one step is required, therefore $a_1 = 1$ let us denote

the tower on which the disks are originally located as A. To effect the transfer, for $n \geq 1$, we first transfer the top (n – 1) disks to a vacant tower say B in the prescribed manner. This involves a_{n-1} steps (or moves). Then we transfer the nth disk to the other vacant tower say C. This involves 1 move (or one step). Lastly we transfer (n – 1) disks from tower B to the tower C, in the prescribed manner. This involves a_{n-1} steps (or moves). Thus the total number of moves involved in the transfer of 'n' disks is $a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1$ for $n \geq 1$ ...(1)

Here k = 2 and f(n) = 1 (or equivalently $a_{n+1} = 2a_n + 1$, for $n \geq 0$)

The recurrence relation is non-homogenous and hence its general solution is given by

$$a_n = k^n a_0 + \sum_{r=1}^{n} k^{n-r} f(r)$$

i.e., $a_n = 2^n(0) + \sum_{r=1}^{n} 2^{n-r}(1)$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$$

$a_n = \dfrac{2^n - 1}{2 - 1} = 2^n - 1$ is the required number of moves. This is the solution of tower of Hanoi problem.

EXERCISE:

I. Solve the following recurrence relations:

1. $3a_{n+1} - 4a_n = 0, n \geq 0, a_1 = 5$
2. $2a_n = 3a_{n-1}, n \geq 1, a_4 = 81$
3. $a_n = a_{n-1} + n, n \geq 2, a_1 = 7$
4. $a_n = 2a_{n-1} + 1, n \geq 2, a_1 = 7$
5. $a_{n+1} = a_n + (3n^2 - n), n \geq 0, a_0 = 3$
6. $a_n - a_{n-1} = 3n^2, n \geq 1, a_0 = 7$
7. $a_{n+1} - 2a_n = 2^n, n \geq 0, a_0 = 1$
8. $a_{n+1} = 2a_n + 5, n \geq 0, a_0 = 1$
9. $a_n = a_{n-1} + \dfrac{1}{n(n+1)}, n \geq 1, a_0 = 1$

II. Find the recurrence relation, initial condition and the general solution for each of the following sequences:

1. $6, -18, 54, -162, \ldots$
2. $7, \frac{14}{5}, \frac{28}{25}, \frac{56}{125}, \ldots$
3. $8, \frac{24}{7}, \frac{72}{49}, \frac{216}{343}, \ldots$
4. 3, 7, 11, 15,...

ANSWERS:

I.

1. $\left(\frac{15}{4}\right)\left(\frac{4}{3}\right)^n$ 2. $\frac{3^n}{2^{n-4}}$ 3. $3+\frac{n(n+1)}{2}$

4. $4(2^n)-1$ 5. $3+n(n-1)^2$ 6. $(n+1)^3$

7. $2^n+(n\,2^{n-1})$ 8. $6(2^n)-5$ 9. $2-\frac{1}{(n+1)}$

II.

1. $a_n=-3a_{n-1}\,(n\ge1),\ a_0=6$ 2. $a_n=\left(\frac{2}{5}\right)a_{n-1}\,(n\ge1),\ a_0=7$

3. $a_n=\left(\frac{3}{7}\right)a_{n-1},\,(n\ge1),\ a_0=8$ 4. $a_n=a_{n-1}+4,\,(n\ge1),\ a_0=3$

2.12 Second-order linear homogeneous recurrence relation with constant coefficients:

A recurrence relation of the form

$C_na_n+C_{n-1}a_{n-1}+C_{n-2}a_{n-2}=0$ for $n\ge2$ ...(1)

Where C_n, C_{n-1}, C_{n-2} are real constants with $C_n\ne0$ is called a **second order linear homogenous recurrence relation with constant coefficients.**

We seek a solution of relation (1) in the form $a_n=Ck^n$ $(C\ne0,\ k\ne0)$ ...(2)

Using (2) in (1) we get

$C_nCk^n+C_{n-1}Ck^{n-1}+C_{n-2}Ck^{n-2}=0$

or $Ck^{n-2}(C_{n-1}k^2+C_{n-1}k+C_{n-2})=0$

$\Rightarrow C_nk^2+C_{n-1}k+C_{n-2}=0$ ...(3)

(3) is called the characteristic equation of (1) Since (3) is a quadratic equation is k; it has two roots which can be

(i) Real and distinct

(ii) Real and coincident

(iii) Complex roots which are conjugates of each other.

Case (i): Suppose the roots are real and distinct

Let k_1, k_2 be the two roots, then the general solution of (1) is of the form $a_n=c_1k_1^n+c_2k_2^n$ ...(4)

Where c_1 and c_2 are arbitrary real constants

Case (ii): Suppose the roots are real and coincident

Let the two roots k_1 and k_2 are real and equal say k is the common value then the general solution of (1) is of the form $a_n=(c_1+c_2n)\,k^n$ ...(4)

where c_1 and c_2 are arbitrary real constants

Case (iii): Suppose the roots are complex:

Let the two roots k_1 and k_2 are complex. Then they are complex conjugates of each other, so that if $k_1=\alpha+i\beta$ then $k_2=\alpha-i\beta$. Then the general solution of (1) is of the form

$a_n=r^n\,(c_1\cos n\theta+c_2\sin n\theta)$

where $r=\sqrt{\alpha^2+\beta^2}$ and $\theta=\tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Further if a_0 and a_1 are specified, the specified values are called the **'initial conditions'**. If these conditions are applied to the general solution, then we get the values of the arbitrary constants c_1 and c_2; which is called the **particular solution of** (1).

[Case (iii) detail explanation:

$k_1=(\alpha+i\beta),\ k_2=(\alpha-i\beta)$

$a_n=Ak_1^n+Bk_2^n=A(\alpha+i\beta)^n+B(\alpha-i\beta)^n$

Where A and B are constants

$\alpha\pm i\beta=r(\cos\theta\pm i\sin\theta)$

$\Rightarrow \alpha=r\cos\theta,\ \beta=r\sin\theta$

$\alpha^2+\beta^2=r^2\Rightarrow r=\sqrt{\alpha^2+\beta^2}$

and $\frac{\beta}{\alpha}=\frac{r\sin\theta}{r\cos\theta}\Rightarrow\frac{p}{\alpha}=\tan\theta$

or $\theta=\tan^{-1}\frac{\beta}{\alpha}$

$a_n=A\left[\{r(\cos\theta+i\sin\theta)\}^n\right]+B\left[\{r(\cos\theta-i\sin\theta)\}^n\right]$

using De'Movire's theorem

$$= A\left[r^n(\cos n\theta + i\sin n\theta)\right] + B\left[r^n(\cos n\theta - i\sin n\theta)\right]$$

$$= r^n\left[(A\cos n\theta + B\cos n\theta) + i(A\sin n\theta - B\sin n\theta)\right]$$

$$= r^n\left[\cos n\theta(A+B) + i\sin n\theta(A-B)\right]$$

$$a_n = r^n\left[c_1\cos n\theta + c_2\sin n\theta\right]$$

where $A + B = c_1$ and $i(A - B) = c_2$

WORKED EXAMPLES:

Example 1: Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 0, n \geq 2, a_0 = \frac{5}{2}, a_1 = 8$

Solution:

We have $a_n - 4a_{n-1} + 4a_{n-2} = 0$

The corresponding characteristic equation is $k^2 - 4k + 4 = 0 \Rightarrow (k-2)^2 = 0 \Rightarrow k = 2, 2$

$\therefore$ General solution: $a_n = (c_1 + c_2 n)2^n$. ...(1)

Using the initial conditions $a_0 = \frac{5}{2}, a_1 = 8$

Put n = 0 in (1) we get

$a_0 = (c_1 + c_2 \cdot 0)2^0 = c_1$

i.e., $c_1 = \frac{5}{2}$

punt n = 1 in (1) we get

$a_1 = (c_1 + c_2)\, 2 = \left(\frac{5}{2} + c_2\right)2$

$8 = 5 + 2c_2 \Rightarrow 3 = 2c_2$ or $c_2 = \frac{3}{2}$

Substituting these values in (1) we get

$a_n = \left(\frac{5}{2} + \frac{3}{2}n\right)2^n$ which is the particular solution.

Example 2: Solve $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = -1$ and $a_1 = 8$

Solution:

We have, $a_n + a_{n-1} - 6a_{n-2} = 0$

The characteristic equation is

$k^2 + k - 6 = 0$ or $(k+3)(k-2) = 0$

$\Rightarrow k = 2, -3$ which are real and distinct.

Therefore the general solution of the given recurrence relation is

$a_n = c_1 2^n + c_2(-3)^n$ ...(1)

where c_1 and c_2 are arbitrary constants. Now using the initial conditions

$a_0 = -1$ and $a_1 = 8$ in (1)

for n = 0, $a_0 = c_1 + c_2$

$\Rightarrow -1 = c_1 + c_2$ or $c_1 + c_2 = -1$ ...(2)

For n = 1, $a_1 = c_1 . 2 + c_2(-3)$

$\Rightarrow 8 = 2c_1 - 3c_2$ or $2c_1 - 3c_2 = 8$ ...(3)

Solving (2) and (3)

$c_1 + c_2 = -1 \times 2 \quad 2c_1 + 2c_2 = -2$

$2c_1 - 3c_2 = 8 \quad 2c_1 - 3c_2 = 8$

$5c_2 = -10$ or $c_2 = -2$

Now $c_1 + c_2 = -1$

$c_1 - 2 = -1$ or $c_1 = 2 - 1 = 1$

Thus $c_1 = 1$ and $c_2 = 2$ substituting these values in (1) we get

$a_n = 2^n - 2(-3)^n$ which is the particular solution.

Example 3: Solve $a_n - a_{n-1} + a_{n-2} = 0$

Solution:

The characteristic equation is $(k^2 - k + 1) = 0 \Rightarrow k = \frac{1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2}$

i.e., $k = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Here $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$.

The roots of the characteristic equation are complex.

Hence the general solution is given by

$a_n = r^n(c_1\cos n\theta + c_2\sin n\theta)$

where $r=\sqrt{\alpha^2+\beta^2}$ and $\theta=\tan^{-1}\dfrac{\beta}{\alpha}$

i.e., $r=\sqrt{\dfrac{1}{4}+\dfrac{3}{4}}=1;\ \theta=\tan^{-1}\sqrt{3}=\dfrac{\pi}{3}$

$\therefore\ a_n=\left(c_1\cos\dfrac{n\pi}{3}+c_2\sin\dfrac{n\pi}{3}\right)$

Example 4: Solve $a_n = 2(a_{n-1} - a_{n-2}),\ n \geq 2$ given that $a_0 = 1$ and $a_1 = 2$.

Solution:

We have, $a_n - 2a_{n-1} + 2a_{n-2} = 0,\ (n \geq 2)$

The corresponding characteristic equation is $k^2 - 2k + 2 = 0$ whose roots are

$$k=\frac{2\pm\sqrt{4-4\times1\times2}}{2}=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm2i}{2}$$

i.e., $k=1\pm i$ where $\alpha=1$ and $\beta=1$

$r=\sqrt{\alpha^2+\beta^2}=\sqrt{1+1}=\sqrt{2}$

$\theta=\tan^{-1}\dfrac{\beta}{\alpha}=\tan^{-1}\dfrac{1}{1}=\tan^{-1}1=\dfrac{\pi}{4}$

Hence the general solution is

$$a_n=\left(\sqrt{2}\right)^n\left[c_1\cos\frac{n\pi}{4}+c_2\sin\frac{n\pi}{4}\right]\quad\ldots(1)$$

Using the given initial conditions $a_0 = 1$ and $a_1 = 2$

For $n = 0$, $a_0=\left(\sqrt{2}\right)^0\left[c_1\cos0+c_2\sin0\right]$

$1 = c_1$ or $c_1 = 1$

For $n = 1$, $a_1=\left(\sqrt{2}\right)^1\left[c_1\cos\dfrac{\pi}{4}+c_2\sin\dfrac{\pi}{4}\right]$

$2=\sqrt{2}\left[c_1\dfrac{1}{\sqrt{2}}+c_2\cdot\dfrac{1}{\sqrt{2}}\right]$

$2=c_1+c_2$ we have $c_1 = 1$ $\qquad\therefore 2 = 1 + c_2$ or $c_2 = 1$

Substituting the values of c_1 and c_2 in (1)

we get $a_n=\left(\sqrt{2}\right)^n\left[\cos\dfrac{n\pi}{4}+\sin\dfrac{n\pi}{4}\right]$

which is the particular solution

2.13 Fibonacci numbers:

A special sequence of numbers called Fibonacci numbers whose recursive definition is given by,

$F_n=F_{n-1}+F_{n-2},\ n\geq2$ and $F_0=0,\ F_1=1$

Fibonacci sequence $\{F_n\}$:

We shall find out a few terms of the sequence $\{F_n\}$, $n \geq 2$ starting from the recursive definition of F_n.

$F_n=F_{n-1}+F_{n-2};\ F_0=0,\ F_1=1$

$F_2=F_1+F_0=1+0=1,\quad F_3=F_2+F_1=1+1=2$

$F_4=F_3+F_2=2+1=3,\ F_5=F_4+F_3=8+5=13$

$F_8=F_7+F_6=13+8=21,\ F_9=F_8+F_7=21+13=34$

Hence Fibonacci sequence is given by $\{F_n\} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Example 5: Solve Fibonacci relation $F_{n+2} = F_{n+1} + F_n,\ n \geq 0,\ F_0 = 1,\ F_1 = 1$

Solution:

The given Fibonacci relation is equivalent to $F_n = F_{n-1} + F_{n-2}\ n \geq 0$

i.e., $F_n-F_{n-1}-F_{n-2}=0$. The corresponding characteristic equation is $k^2 - k - 1 = 0$

$$\Rightarrow k=\frac{1\pm\sqrt{1-4\times1\times(-1)}}{2}=\frac{1\pm\sqrt{5}}{2}$$

$\therefore$ General solution is

$$F_n=c_1\left(\frac{1+\sqrt{5}}{2}\right)^n+c_2\left(\frac{1-\sqrt{5}}{2}\right)^n\quad\ldots(1)$$

Now using the initial conditions $F_0 = 1,\ F_1 = 1$

For $n = 0$, $F_0 = c_1 + c_2 \Rightarrow 1 = c_1 + c_2$ or $c_1 + c_2 = 1$ $\qquad\ldots(2)$

For $n = 1$, $F_1=c_1\left(\dfrac{1+\sqrt{5}}{2}\right)+c_2\left(\dfrac{1-\sqrt{5}}{2}\right)$

$\Rightarrow\quad 1=c_1\left(\dfrac{1+\sqrt{5}}{2}\right)+c_2\left(\dfrac{1-\sqrt{5}}{2}\right)$

or $\left(1+\sqrt{5}\right)c_1+\left(1-\sqrt{5}\right)c_2=2$ $\qquad\ldots(3)$

Solving (2) and (3) i.e., use $c_2 = -c_1$ in (3)

we get $(1+\sqrt{5})c_1 - (1-\sqrt{5})c_1 = 2$

$$c_1 + \sqrt{5}c_1 - c_1 + \sqrt{5}c_1 = 2$$

$$\Rightarrow 2\sqrt{5}c_1 = 2 \text{ or } c_1 = \frac{1}{\sqrt{5}}$$

$$\therefore c_2 = -\frac{1}{\sqrt{5}}$$

Substituting these values in the general solution (1), we get

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Which is the particular solution.

EXERCISE:

Solve the following recurrence relations:

1. $a_{n+2} = 4a_{n+1} - 4a_n;\ n \geq 0,\ a_0 = 1, a_1 = 3$
2. $a_n = 5a_{n-1} + 6a_{n-2};\ n \geq 2,\ a_0 = 1,\ a_1 = 3$
3. $2a_n = 7a_{n-1} - 3a_{n-2};\ n \geq 2, a_0 = 2,\ a_1 = 5$
4. $a_n - 6a_{n-1} + 9a_{n-2} = 0;\ n \geq 2,\ a_0 = 5,\ a_1 = 12$
5. $a_n + a_{n-1} - 6a_{n-2} = 0;\ n \geq 2,\ a_0 = -1, a_1 = 8$
6. $a_n + a_{n-1} - 6a_{n-2} = 0$
7. $a_n - 5a_{n-1} + 6a_{n-2} = 0$
8. $2a_{n+2} - 11a_{n+1} + 5a_n = 0,\ n \geq 0,\ a_0 = 2,\ a_1 = -8$
9. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$
10. $a_n + 7a_{n-1} + 8a_{n-2} = 0$ for $n \geq 2,\ a_0 = 2,\ a_1 = -7$

ANSWERS:

1. $a_n = 2^n + n2^{n-1}$
2. $a_n = \left(\frac{3}{7}\right)(-1)^n + \left(\frac{4}{7}\right)6^n$
3. $a_n = \left(\frac{1}{5}\right)(8 \cdot 3^n + 2^{1-n})$
4. $a_n = (5-n)3^n$
5. $a_n = (-2)(-3)^n + 2^n$
6. $a_n = c_1(-3)^n + c_2 2^n$
7. $a_n = c_1 3^n + c_2 2^n$
8. $a_n = 4\left(\frac{1}{2}\right)^n - 2(5^n)$
9. $a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$
10. $a_n = \left(\frac{-7+\sqrt{17}}{2}\right)^n + \left(\frac{-7-\sqrt{17}}{2}\right)^n$

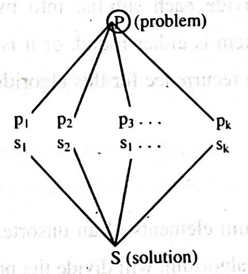
2.14 Divide and Conquer Relations:

Divide and conquer algorithm consists of a dispute using the following there steps. **Divide** the original problem in to a set of sub-problems.

Conquer: Solve every sub problem individually recursively.

Combine: Put together the solutions of the sub-problems to get the solution of the whole problem.

1. **Divide:** Break the given problem into sub-problems of the same type. This step involves breaking the problem in to smaller sub-problems. Sub-problems should represent a part of the original problem. This step generally takes a recursive approach to divide the problem until no sub-problem is further divisible. At this stage, sub-problems become atomic in nature but still represent some part of the actual problem.
2. **Conquer:** Recursively solve these sub-problems. This step receives a lot of smaller sub-problems to be solved. Generally at this level the problems are considered solved on their own.
3. **Combine:** Appropriately combine the answer. When the smaller sub-problems are solved, this stage recursively combines them until they formulate a solution to the original problem.



P: Main problem

$P_1, P_2, P_3, \ldots, P_k$: sub problems

$S_1, S_2, S_3, \ldots, S_k$: sub solutions

S: solution to the main problem.

Examples:

1. **Computational complexity:**

   The computational complexity of a divide and conquer algorithm can be estimated by using a mathematical formula known as a **'recurrence relation'**. If we have a problem of size n, then suppose the recursive algorithm divides the problem in to a sub-problems; each of size $\left(\frac{n}{b}\right)$. Additionally, suppose that g(n) represents any further computations that are needed to combined the solutions of the sub-problems into the overall solution of the original problem. If f(n) represents the **'number of operations'** needed to solve a problem of size n, then f(n) can be written as the following recurrence relation

$$f(n) = a\ f\left(\frac{n}{b}\right) + g(n)$$

This is the type of recurrence relation that we will solve to find the complexity of divide and conquer algorithms.

2. **Binary search:**
Consider a sorted list of n items. Suppose we want to search for an item in that list. A divide-and-conquer approach involves finding the middle item and dividing the list in to two sub-lists, a left sub-list and a right sub-list. By comparing the item of interest to the middle item of the list, it is easy to determine whether the item is in the left sub-list or the right sub-list. Once it is known which sub-list the item may be in, the other sub-list is no longer considered. The algorithm continues to divide each sub-list into two until the sub-lists contain only one item. At this point, the item is either found, or it is determined that the item is not in the list. In order to develope a recurrence for this algorithm, we need to know.

3. **Finding the Maximum and Minimum of a list:**
Suppose we want to find the maximum and minimum elements of an unsorted list of 'n' items. Similar to binary search a divide and conquer algorithm will divide the problem in to two sub-lists of half size and then recursively find the maximum and minimum elements of these sub-lists. In addition two more comparisons are performed: one to compare the maximum element of the two sub-lists and one to compare the minimum element of the two sub-lists. If f(n) is the number of comparisons performed (with 'n' even) the resulting recurrence relation is $f(n) = 2f\left(\frac{n}{2}\right) + 2$

2.15 Representing relations using matrices and digraphs:

Introduction: This section is in continuation of **'relations and functions'** discussed earlier. Special types of relations called **'equivalence relations'** is studied in this section in addition, matrix representation of relations and pictorial representation of relations known as **'digraphs'** are studied. Matrix and graphical presentation of relation is highly useful in visualzing several concepts of computer applications.

Matrix representation of a relation-zero-one Matrices and Directed graphs:

Let A = $\{a_1, a_2, a_3, \ldots, a_m\}$ and B = $\{b_1, b_2, b_3, \ldots, b_n\}$ be two finite sets of orders m and n respectively. Then the Cartesian product of A and B denoted by A × B is given by

$$A \times B = \{(a_i, b_j) \mid 1 \le i \le m, 1 \le j \le n\}$$

This is represented in the form of on m × n matrix as follows:

$$\begin{bmatrix} (a_1,b_1) & (a_1,b_2) & (a_1,b_3) & \cdots & (a_1,b_n) \\ (a_2,b_1) & (a_2,b_2) & (a_2,b_3) & \cdots & (a_2,b_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a_m,b_1) & (a_m,b_2) & (a_m,b_3) & \cdots & (a_m,b_n) \end{bmatrix}$$

Let R be a relation from A to B so that R is a subset of A × B Matrix representation of a relation R denoted by M(R) is defined as follows.

$$M(R) = [m_{ij}] \text{ where } m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

m_{ij} represents the element in the ith row and jth column of the matrix.

Since M(R) contains only 0 and 1 elements, it is called the **'zero-one matrix'**.

Note:

1. Rows of M(R) correspond to the elements of A and the columns correspond to the elements of B.
2. When B = A, the matrix M(R) becomes an n × n matrix whose elements are $m_{ij} = (a_i, a_j)$ with $m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, a_j) \notin R \end{cases}$

Illustrative Examples:

Example 1: Let A = {0, 1, 2} and B = {a, b} and the relation R from A to B defined by R = {(0, a), (1, b), (2, a)}

Solution:

Here $A = \{a_1, a_2, a_3\} = \{0, 1, 2\}$

$B = \{b_1, b_2\} = \{a, b\}$

Note that $m_{11} = (a_1, b_1) = (0, a)$ because $(0, a) \in R$

$m_{12} = (a_1, b_2) = (0, b)$ because $(0, b) \notin R$

$m_{21} = (a_2, b_1) = (1, p) = 0$

$m_{22} = (a_2, b_2) = (1, q) = 1$

$m_{31} = (a_3, b_1) = (2, p) = 1$

$m_{32} = (a_3, b_2) = (2, q) = 0$

Accordingly the matrix of the relation R is

$$M(R)=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}$$

Example 2: Let A = {a_1, a_2}, B = {b_1, b_2}

Solution:

Then we have $A\times B=\{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2)\}$

Consider the following relations,

$R_1=\{(a_1,b_1),(a_2,b_2)\}$, $R_2=\{(a_1,b_2),(a_2b_1)\}$

$R_3=\{(a_1,b_1),(a_1,b_2),(a_2,b_2)\}$

We have the following matrix representations

$$M(A\times B)=\begin{bmatrix}(a_1,b_1) & (a_2,b_2)\\ (a_2,b_1) & (a_2,b_2)\end{bmatrix}$$

$$M(R_1)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};M(R_2)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$M(R_3)=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

Example 3: Let A = {a, b, c} and B = {1, 2}

Solution:

$A\times B=\{(a,1),(a,2),(b,1)(b,2)(c,1)(c,2)\}$

$R_1=\{(a,1),(b,2),(c,1)\}$, $R_2=\{(a,2),(b,2)(c,2)\}$

$R_3=\{(a,1),(b,1),(c,1),(c,2)\}$

$$M(A\times B)=\begin{bmatrix}(a,1) & (a,2)\\ (b,1) & (b,2)\\ (c,1) & (c,2)\end{bmatrix}$$

$$\therefore\ M(R_1)=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix};M(R_2)=\begin{bmatrix}0 & 1\\ 0 & 1\\ 0 & 1\end{bmatrix};M(R_3)=\begin{bmatrix}1 & 0\\ 1 & 0\\ 1 & 1\end{bmatrix}$$

Example 4: Let $M(R)=\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{bmatrix}$

Solution:

We can write the associated relation R.

M(R) is a 3 × 4 matrix and hence take

$A=\{a_1, a_2, a_3\}$ and $B=\{b_1,b_2,b_3,b_4\}$

$\therefore\ R=\{(a_1,b_1),(a_1,b_3),(a_1,b_4),(a_2,b_2),(a_2,b_4),(a_3,b_1),(a_3,b_3)\}$

2.16 Graphical representation of a relation:

Directed graph or Digraph:

Let R be a relation on a finite set $A(R\subseteq A\times A)$. Then R can be represented pictorially as described below.

Step 1:

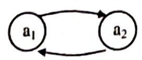
(i) Draw a small circle for every element of the set A and the element is written within the circle. These circles are called **Vertices** or **nodes.** (if it is, one circle then it is called a **vertex** or **node**)

Step 2:

(i) Draw an **arrow** called an **edge** from a **vertex** a_1 to a **vertex** a_2 if and only if $(a_1, a_2) \in R$

This pictorial representation of R is called a **directed graph** or **diagraph** of R.

(ii) If (a_2, a_1) also belongs to R, another arc is drawn between the two with appropriate indication of direction. Sometimes we use the same arc with pointed row in both the directions. So for $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$. The digraph is as shown below.



or

(iii) For $(a_1, a_1) \in R$, we draw a small arc which starts from one vertex and returns there only. Such an arc is called a **'loop'**

For $(a_1, a_2) \in R$, $(a_2, a_1) \in R$, $(a_1, a_1) \in R$, $(a_2, a_2) \in R$ the digraph is as shown below

(iv) If a relation is pictorially represented by a digraph, a vertex from which an edge leaves is called the **'origin'** or the **'source'** for that edge and a vertex where an edge ends is called the **'terminus'** of that edge.

(v) A vertex which is neither a **source** nor a **terminus** of any edge is called an **Isolated vertex.**

(vi) An edge for which the source and the terminus are one and the same vertex is called a **loop.** (which is already explained above)

Note: Alternative notation for the digraph.

(i) V = set of all vertices, E = edges (arrows) corresponding to vertices, Digraph G = (V, E)

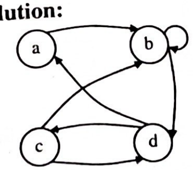
(ii) The number of arrows terminating at a vertex is called the **In degree** of that vertex denoted by I_d. This is equal to the number of right co-ordinates of the vertex in the ordered pairs of R.

(iii) The number of arrows leaving a vertex is called the **out degree** of that vertex denoted by O_d. This is equal to the number of left co-ordinates of the vertex in the ordered pairs of R.

Illustrative Examples:

Example 1: Draw the digraph of the relation R = {(a, b), (b, b), (b, d) (c, b), (c, d), (d, a), (d, c)} defined on the set A = {a, b, c, d}

Solution:

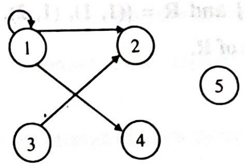


We observe that there is a loop at the vertex b. In degrees of the vertices a, b, c, d are 1, 3, 1, 2 respectively. Further out degrees of a, b, c, d are 1, 2, 2, 2 respectively.

Example 2: Let A = {1, 2, 3, 4, 5} and the relation R = {(1, 1), (1, 2), (1, 4), (3, 2)} defined on A. Draw the digraph of R.

Solution:

The digraph is as shown below



We observe that vertex 1 has a loop and the vertex 5 is an isolated vertex.

Indegrees of the vertices 1, 2, 3, 4 are 1, 2, 0, 1 respectively and the outdegree of the vertices 1, 2, 3, 4 are 3, 0, 1, 0 respectively.

WORKED EXAMPLES:

Example 1: Let A = {a, b, c} and B = {0, 1} and R = {(a, 0), (b, 0) (c, 1) be the relation from A to B. Write down the matrix of this relation.

Solution:

We have A = {a, b, c} and B = {0, 1}. $A \times B = \{(a,0),(a,1),(b,0),(b,1),(c,0),(c,1)\}$.

$(a,0) \in R, (a,1) \notin R, (b,0) \in R, (b,1) \notin R\ (c,0) \notin R$ and $(c,1) \in R$.

$$\therefore\ M(R) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2: Let A = {1, 2, 3, 4} and R be the relation on A defined by (a, b) ∈ R if and only if a ≤ b. Write down R as a set of ordered pairs. Also write down the matrix of this relation.

Solution:

Given $A = \{1,2,3,4\}$

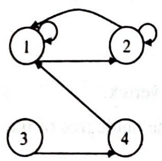
$\therefore\ A\times A = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$
$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

Now $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)(3,3),(3,4),(4,4)\}$

$$M(R) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3: Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1)} be a relation on A. Draw the digraph of R.

Solution:



Example 4: Let A = {1, 2, 3, 4} and let R be the relation on A defined by aRb if and only if b = 2a.

Solution:

(i) Write down 'R' as a set of ordered pairs

(ii) Draw the digraph of R.

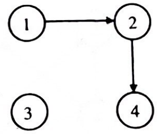
(iii) Determine the in-degrees and out-degrees of the vertices in the digraph.

Solution:

Given $A = \{1,2,3,4\}$ using a R b defined by b = 2a, we write

(i) R = {(1, 2), (2, 4)}

(ii) Digraph:



(iii) From the digraph, we observe that 3 is an isolated vertex. The indegrees and outdegrees are as shown in the following table.

Vertex	1	2	4
Indegree	0	1	1
Outdegree	1	1	0

Example 5: Let A = {a, b, c, d}. R is defined as
$R = \{(a,a),(a,b),(b,a),(b,b),(b,c),(b,d),(c,d),(d,a),(d,d)\}$

(i) Write down the matrix representation of the relation R.

(ii) Draw the digraph of R

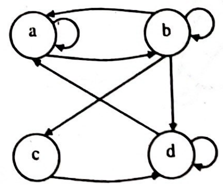
(iii) Determine the in-degree and out-degree of each vertex or node.

Solution:

We have A = {a, b, c, d}

$$M(A\times A) = \begin{bmatrix} (a,a) & (a,b) & (a,c) & (a,d) \\ (b,a) & (b,b) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,c) & (c,d) \\ (d,a) & (d,b) & (d,c) & (d,d) \end{bmatrix} \quad (i)\ \therefore M(R) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Digraph of R:



(iii)

Vertex	a	b	c	d
Indegree	3	2	1	3
Outdegree	2	4	1	2

Example 6: Let A = {1, 2, 3, 4} and let R be the relation on A defined by aRb if and only if "a divides b",

(i) Write down R as a set of ordered pairs

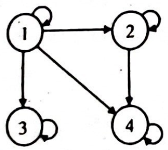
(ii) Draw the digraph of R

(iii) Determine the in-degrees and out-degrees of the vertices in the digraph.

Solution

(i) $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

(ii) Digraph of R:



(iii)

Vertex	1	2	3	4
Indegree	1	2	2	3
Outdegree	4	2	1	1

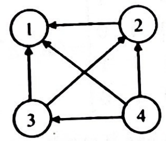
Example 7: If R = {(x, y) | x > y} is a relation defined on the set A = {1, 2, 3, 4}, write down the matrix and the digraph of R.

Solution:

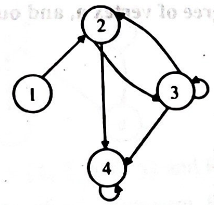
$R = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

$$M(R) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Digraph of R:



Example 8: Find the relation R determined by the digraph given below. Also write the matrix of the relation.

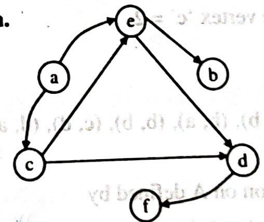


Solution:

$R = \{(1,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,4)\}$

$$M(R) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 9: Find the relation R determined by the digraph given below. Also write the matrix of the relation.



Solution:

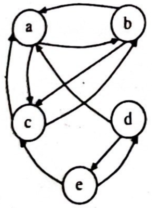
$R = \{(a,c),(a,e),(c,d),(c,e),(d,f),(e,b)(e,d)\}$

$$M(R)=\begin{bmatrix}0&0&1&0&1&0\\0&0&0&0&0&0\\0&0&0&1&1&0\\0&0&0&0&0&1\\0&1&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}$$

Example 10: Draw the digraph G= (V, E) given that V = {a, b, c, d, e}, E = {(a, b), (a, c), (b, a), (b, c), (c, a), (c, b), (d, a), (d, e), (e, c), (e, d)}

Also write the matrix of the relation set E. Write the in degree of vertex a, and out degree of vertex a.

Solution:



$$M(E)=\begin{bmatrix}0&1&1&0&0\\1&0&1&0&0\\1&1&0&0&0\\1&0&0&0&1\\0&0&1&1&0\end{bmatrix}$$

In degree of vertex 'a' = 3; out degree of the vertex 'c' = 2

EXERCISE:

1. Let A = {a, b, c, d} and R = {(a, a), (a, b), (b, a), (b, b), (c, d), (d, a)} be a relation on A. Write down the digraph of R.
2. Let $A=\{1,2,3,4,5,6\}$ and R be the relation on A defined by $R=\{(1,2),(1,4),(2,3),(2,5),(4,2),(4,5),(5,3),(5,6),(6,4)\}$ Draw the digraph of R.

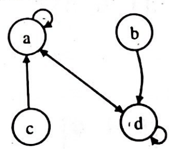
3. Let A = {1, 2, 3, 4} and R be the relation on A that has matrix

$$M(R)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\0&1&0&1\end{bmatrix}$$

(i) Write R and draw the digraph of R

(ii) Write down the in degrees and out degrees of all the vertices.

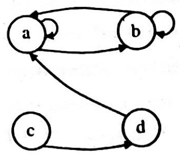
4. Find the relation R from the following directed graph of R. Obtain the in degrees and out degrees of each vertex. Also write the M(R).



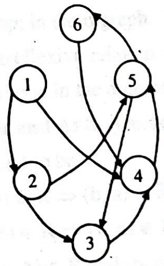
5. Let A = {1, 2, 3, 4, 6} and R be the relation on A defined by x R y if and only if "x is a multiple of y". Represent R as a set of ordered pairs. Draw the digraph and matrix representation of R.

ANSWERS:

1.

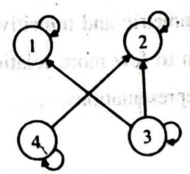


2.



3. (i) $R=\{(1,1),(2,2),(3,1),(3,2),(3,3),(4,2),(4,4)\}$

Digraph of R:



(ii)

Vertex	1	2	3	4
In-degree	2	3	1	1
Out-degree	1	1	3	2

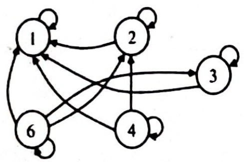
4. $R=\{(a,a),(a,d),(b,d),(c,a),(d,a),(d,d)\}$

Vertex	a	b	c	d
In-degree	3	0	0	3
Out-degree	2	1	1	2

$$M(R)=\begin{bmatrix}1&0&0&1\\0&0&0&1\\1&0&0&0\\1&0&0&1\end{bmatrix}$$

5. $R=\{(1,1),(2,1),(2,2),(3,1),(3,3),(4,1),(4,2),(4,4),(6,1),(6,2),(6,3),(6,6)\}$

Digraph of R:



$$M(R)=\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&0&1&0&0\\1&1&0&1&0\\1&1&1&0&1\end{bmatrix}$$

2.17 Types of Relations

We have earlier studied types of Relations namely Reflexive, Symmetric and transitive under unit I, now in this section we study the same relations in addition to few more relations and discuss all the relations in the perspective of matrix and graphical representations.

1. **Reflexive and Irreflexive relations:**

A relation R on a set A is said to be **Reflexive** if $(a, a) \in R \; \forall a \in A$.

A relation R on a set A is said to be irreflexive if $(a, a) \notin R \; \forall a \in A$.

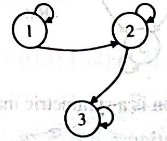
Example: Let A = {1, 2, 3}

$R_1=\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$

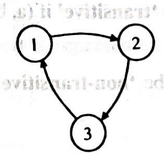
$R_2=\{(1,2),(2,3),(3,1)\}$

R_1 is reflexive, where as R_2 is Irreflexive

$M(R_1)=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$; digraph:



$M(R_2)=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$; Digraph:



Note:

(i) In the matrix of a reflexive relation, all the principal diagonal elements are equal to 1 every vertex has a loop; in the digraph.

(ii) In the matrix of an irreflexive relation all the principal diagonal elements are equal to zero. No vertex has a loop in the digraph.

2. **Symmetric, Asymmetric and Anti-symmetric relations;**

A relation on a set A is said to be

(i) **'Symmetric'** if $(a, b) \in R \Rightarrow (b, a) \in R \; \forall a, b \in A$

(ii) **'Asymeetric'** if $(a, b) \in R \Rightarrow (b, a) \notin R \; \forall a, b \in A$

(iii) **Anti symmetric** if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b \; \forall a, b \in A$

Example:

Let A = {1, 2, 3}

$R_1=\{(1,1),(1,2),(2,1),(1,3),(3,1)\}$ symmetric

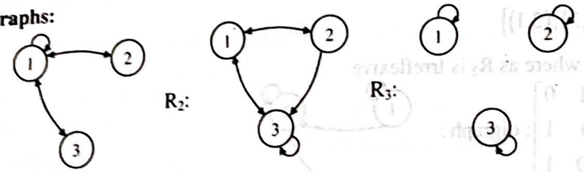
$R_2=\{(1,2),(2,3),(3,1),(3,3)\}$ Asymmetric

$R_3=\{(1,1),(2,2),(3,3)\}$ Anti symmetric

Matrix representation:

$$M(R_1)=\begin{bmatrix}1&1&1\\1&0&0\\1&0&0\end{bmatrix};M(R_2)=\begin{bmatrix}0&1&0\\0&0&1\\1&0&1\end{bmatrix};M(R_3)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Digraphs:



Note: Matrix of a symmetric relation is a symmetric matrix

3. **Transitive and non-transitive relations:**

A relation R on a set A is said to be **'transitive'** if (a, b) ∈ R and (b, c) ∈ R then (a, c) ∈ R ∀ a, b, c ∈ A

A relation R on a set A is said to be **'non-transitive'** if (a, b) ∈ R and (b, c) ∈ R then **(a, c) ∉ R** ∀ a, b, c ∈ A.

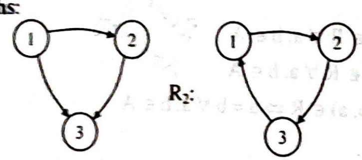
Example: Let A = {1, 2, 3}

$R_1=\{(1,2),(2,3),(1,3)\}$ Transitive

$R_2=\{(1,2),(2,3),(3,1)\}$ Non-transitive

$$M(R_1)=\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix};M(R_2)=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

Digraphs:

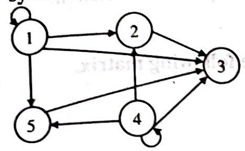


Equivalence Relation:

A relation R on a set A is said to be an equivalence relation if it is **reflexive, symmetric and transitive.**

WORKED EXAMPLES:

Example 1: Verify that the relation represented by the following digraph is anti symmetric and transitive.



Solution:

From the given digraph, we have $R=\{(1,1),(1,2),(1,3),(1,5),(2,3),(4,4),(4,2),(4,5),(5,3)\}$

R is anti symmetric because (1, 2) ∈ R but (2, 1) ∉ R.

R is not transitive because (4, 5) ∈ R, (5, 3) ∈ R but (4, 3) ∉ R.

Example 2: Verify that the relation R represented by the following matrix is irreflexive and symmetric.

$$M(R)=\begin{bmatrix}0&1&0&1\\1&0&1&1\\0&1&0&0\\1&1&0&0\end{bmatrix}$$

Solution:

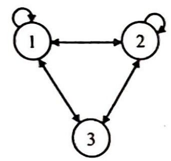
From the given matrix, we have A = {1, 2, 3, 4} and

$R=\{(1,2),(1,4),(2,1),(2,3),(2,4),(3,2),(4,1),(4,2)\}$

R is irreflexive, became (1, 1) ∉ R (2, 2) ∉ R, (3, 3) ∉ R and (4, 4) ∉ R.

R is symmetric because (a, b) ∈ R ⇒ (b, a) ∈ R ∀ a, b ∈ A. This property is satisfied for all the ordered pairs of R.

Example 3: The diagraph of a relation R on the set A = {1, 2, 3}is given below. Examine whether R is an equivalence relation.



Solution:

From the given digraph, we have $R=\{(1,1),(2,2),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}$

We observe that $(3, 3) \notin R$. So R is not reflexive and hence R is not an equivalence relation.

Example 4: A relation R on a set A = {a, b, c} is represented by the following matrix.

$$M(R)=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Examine whether R is an equivalence relation.

Solution:

From the given matrix, we have $R=\{(a,a),(a,c),(b,b),(c,c)\}$

R is not symmetric because $(a, c) \in R$ but $(c, a) \notin R$

Therefore R is not an equivalence relation.

EXERCISE:

1. A relation R an a set A = {1, 2, 3, 4} is represented by the following matrix

$$M(R)=\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

Examine for reflexive, irreflexive, Symmetric, antisymmetric and transitive relations.

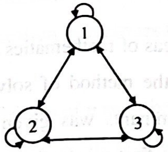
2. A relation R on a set A = {a, b, c, d} is represented by the following matrix

$$M(R)=\begin{bmatrix}1 & 0 & 1 & 1\\ 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$ Examine for equivalence relation.

3. The matrix of the relation R is given below. Show that R is an equivalence relation and draw the digraph of R.

$$M(R)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$

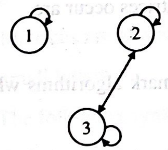
4. A relation R on the set A= {1, 2, 3} is represented by the digraph given below show that R is an equivalence relation.



5. A relation R on the set A = {1, 2, 3, 4} and let R be the relation defined as $R = \{(a, b) \mid a \le b,\ a, b, \in A\}$ show that R is not an equivalence relation and draw the digraph of R.

ANSWERS:

1. Not reflexive, Irreflexive, symmetric, not antisymmetric not transitive.
2. Not an equivalence relation
3. Digraph:



4. Digraph: